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BETTER QUASI-ORDERS FOR UNCOUNTABLE CARDINALS†

BY

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ABSTRACT

We generalize the theory of Nash-Williams on well quasi-orders and better quasi-orders and later results to uncountable cardinals. We find that the first cardinal κ for which some natural quasi-orders are κ-well-ordered, is a (specific) mild large cardinal. Such quasi-orders are $\mathfrak{M}_\lambda$ (the class of orders which are the union of $\leqq \lambda$ scattered orders) ordered by embeddability and the (graph theoretic) trees under embeddings taking edges to edges (rather than to passes).

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§0. Introduction

We call a quasi-order **Q** well-ordered if for every $q_n \in \mathbf{Q}$ $(n < \omega)$, for some $n < m$, $q_n \leqq q_m$. There is a beautiful and extensive theory on this notion, mainly on proving various quasi-orders are well-ordered. See Erdos–Rado [2], Higman [4], Kruskal [7], [8], Rado [21] for the easier parts. For the harder theorems, Nash-Williams suggests a smaller class, that of bqo (better quasi-order) which has a more complicated definition but stronger closure properties. (Important cases are: if **Q** is bqo, so is $\mathrm{Seq}_{<\infty}(\mathbf{Q}) = \{\langle q_i : i < \alpha \rangle : \alpha$ an ordinal, $q_i \in \mathbf{Q}\}$, $\langle q_i : i < \alpha \rangle \leqq \langle q^i : i < \beta \rangle$ iff there is a strictly increasing $h : \alpha \to \beta$, $q_i \leqq q^{h(i)}$ (and similar results on the power set and its iteration); the class of trees of height ω, with the order being: embedding by a function preserving $<$ (and more) but not the level; you can formulate it in graph theoretic terms.) (See [15–20].)

Laver uses this to get deep results: some classes of orders and trees are bqo. But we were initially interested just in trees ordered by such embeddings which preserve the level.[†] Of course this class is not well-ordered, but we want to know whether for every λ it has λ pairwise incomparable elements. A natural approach is to say **Q** is κ-well-ordered iff for every $q_i \in \mathbf{Q}$ $(i < \kappa)$ for some $i < j$, $q_i \leqq q_j$, and to try to generalize Nash-Williams' theory. Notice however that some of the tools are missing: of course the choice of minimal subset like [15], but more important is that the Ramsey theorem becomes problematic (κ has to be weakly compact) and even more Nash-Williams' generalization of it which says "every block contains a barrier" (κ has to be Ramsey).

We succeed in proving the parallel of his basic theorems on "**Q** is bqo iff every $\mathcal{P}_\alpha(\mathbf{Q})$ is well-ordered" for any κ, but we have to change somewhat his basic definitions. In $\mathcal{P}_\alpha(\mathbf{Q})$ we have to assume for $A \subseteq \mathbf{Q}$, $a \in \mathbf{Q}$, that $A \leqq a$ when $(\forall x \in A)\, x \leqq a$ (usually we say $A \not\leqq a$). Alternatively we demand $\mathbf{Q} \times (\omega, \leqq)$ is κ-bqo. Also we lose "every block contains a barrier" so we have to use another intermediate notion κ-I-barrier, and call our notion κ-I-bqo.

We develop this also for a twin notion. We call **Q** κ-narrow if for $q_i \in \mathbf{Q}$

[†] This is interesting because of more general problems in model theory, see [23].

$(i<\kappa)$ for some $i \neq j$, $q_i \leqq q_j$. If we replace consistently κ-well-ordered by κ-narrow, we get κ-D-barriers, κ-D-bqo.

Let the well-ordering number of $\mathbf{Q}$ be the first κ such that $\mathbf{Q}$ is κ-well-ordered.

But if we want to go any further, we have to consider some mildly large cardinal, but don't be afraid if you don't believe in them. The theorems do not say "if some large cardinal exists then..." but rather "the well-ordering cardinal of some naturally defined $\mathbf{Q}$ is a specific large cardinal": so our results are meaningful even if no such cardinal exists.

How are we forced to large cardinals? If $\mathbf{Q}$ is not $\aleph_0$-narrow, and countable, then the first κ for which $\mathbf{Q}$ is κ-I-bqo is the first uncountable beautiful cardinal (see §2, mainly 2.5; it is strongly inaccessible but may exist in L). Also, further theorems require such κ, and a stronger notion — $[\kappa;\lambda]$-I-bqo for any $\aleph_0 \leqq \lambda < \kappa$. Then we get that the trees we mention are $[\kappa;\lambda]$-I-bqo (even when labeled by a $[\kappa;\lambda]$-I-bqo $\mathbf{Q}$), and also $\mathrm{Seq}_{<\infty}(\mathbf{Q})$.

Let $\mathfrak{M}_\lambda$ be the class of ordered sets which are unions of λ scattered orders. (All ordered sets of power $\leqq \lambda$ are inside.) Laver proved that $\mathfrak{M}_{\aleph_0}$ is bqo (under embeddability). Again some of his tools disappear (= the universal (λ,κ)-order of $\mathfrak{M}_\lambda$).

Generally we proved that under general conditions the I and D versions coincide, and the D-well-ordering number is a beautiful cardinal.

But we prove that the well-ordering number of $\mathfrak{M}_\lambda$ is the first beautiful $\kappa > \lambda$.

REMARK. One property we lose when we generalize well-ordering to κ-well-ordering is closure under product of two; this is saved for κ weakly compact. But as for κ-X-bqo we have to make κ Ramsey. If we want to save this property, a reasonable way is as follows. We will have a system $\bar{\mathscr{D}} = \langle \mathscr{D}_B : B \text{ a } \kappa\text{-}X\text{-barrier}\rangle$, $\mathscr{D}_B$ a filter on $B \times B$, and call $\mathbf{Q}$ $\bar{\mathscr{D}}$-X-bqo if for any κ-X-barrier B and $q_\eta \in \mathbf{Q}$ $(\eta \in B)$,

$$\{(\eta,\nu): \eta \in B,\ \nu \in B,\ \eta \mathbf{R}^1_X \nu,\ q_\eta \leqq q_\nu\} \in \mathscr{D}_B.$$

For closure under product of λ, the filters have to be λ^+-complete.

For $\kappa > 2^{\aleph_0}$, a natural filter is: $\mathscr{D}_B$ is generated by the sets $G(B,M)$, M a model with universe κ and say $\aleph_0$ relations, where

$$G(B,M) = \{(\eta,\nu): \eta \in B,\ \nu \in B,\ \eta \mathbf{R}^1_X \nu, \text{ and } \eta, \nu \text{ realize the same type in the model } M \text{ [over the set } \{\alpha : \alpha < \min\{\eta(l), \nu(l) : l\}\}]\}$$

REMARK. Most of the results (mainly in §1) go through for κ an ordinal (infinite for $X = D$, limit for $X = I$). Moreover we have not used the hypothesis that the two place relation $\leqq$ on $\mathbf{Q}$ is a quasi-order. We can define "κ-X-well-ordered", $[\kappa;\alpha]$-X-bqo for any $\mathbf{Q}=(Q,R)$, R a two place relation, as well as, e.g., $\mathscr{P}_\alpha(\mathbf{Q})$. We have not developed this theorem as we have no particular application in mind.

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Review

§0: We define X-barrier (0.1), the depth $\mathrm{Dp}(\eta,B)$, $\mathrm{Dp}(B)$ (0.2) and prove some technical claims to be used later.

§1: We define κ-well-ordered (κ-narrow), B-bqo, (κ,α)-X-bqo, etc. (1.1), product (of quasi-orders) (1.2), $\mathscr{P}(\mathbf{Q})$, $\mathscr{P}_\alpha(\mathbf{Q})$, $\mathscr{P}^*_\alpha(\mathbf{Q})$, $\mathscr{P}^{**}_\alpha(\mathbf{Q})$, $\mathscr{P}^0_\alpha(\mathbf{Q})$ (1.3), characterize order in $\mathscr{P}_{<\infty}(\mathbf{Q})$ (1.7). We prove κ-X-bqo is preserved by $\mathscr{P}$ (1.8), 1.10 is a simple case, 1.9 is used later. By 1.11, $\mathbf{Q}$ is κ-X-bqo iff for every γ $\mathscr{P}_\gamma(\mathbf{Q})$ is κ-X-bqo iff $\mathscr{P}_{<\kappa'}(\mathbf{Q})$ is κ-X-well-ordered. The proof $\mathbf{Q}$ not κ-X-bqo $\rightarrow \mathscr{P}_\alpha(\mathbf{Q})$ not κ-X-well-ordered is carried out in 1.12.

§2: We present weakly compact, Ramsey and beautiful cardinals (2.1–2.4). Important for the paper is 2.5: if $\mathbf{Q}$ is not χ-narrow, then for some α the $\mathscr{D}$-well-ordering number of $\mathscr{P}_\alpha(\mathbf{Q})$ is $\geqq$ the first beautiful $\kappa>\chi$. In 2.7 we define $[\kappa,\alpha;\lambda]$-X-bqo, $[\kappa;\lambda]$-X-bqo, $[\kappa]$-X-bqo and give easy facts (2.8, 2.9). By 2.10, if $|\mathbf{Q}|<\kappa$, κ beautiful then $\mathbf{Q}$ is $[\kappa]$-X-bqo. By 2.11, essentially $\mathbf{Q}$ is $[\kappa;\lambda]$-X-bqo iff $\mathbf{Q}^\lambda$ is κ-X-bqo iff $\mathbf{Q}\times(\lambda,=)$ is κ-X-bqo. In 2.12 we give a trivial sufficient condition for non-κ-narrowness, and by 2.13, $[\kappa;\lambda]$-D-bqo, $[\kappa;\lambda]$-I-bqo are equivalent for $\lambda\geqq\aleph_0$, hence (2.14) the first κ s.t. $\mathbf{Q}$ is $[\kappa;\lambda]$-X-bqo is beautiful. By 2.15 "$[\kappa;\lambda]$-X-bqo" is preserved by $\mathscr{P}^0_\alpha$.

§3: We investigate our notions for $\mathbf{Q}$ linear and get examples showing that $X=D$, $X=I$ may behave very differently (3.1, 3.2); in 3.3–3.5 we get other examples, e.g., for the first beautiful $\kappa>\aleph_0$ we get a κ-I-bqo linear $\mathbf{Q}$ which is not $[\kappa;2]$-I-bqo.

§4: Here we generalize the theorems on preservation of well-ordering (rather than of bqo).

We define $\mathscr{P}^1_{<\kappa}(\mathbf{Q})$ (4.1), prove (4.2) that for κ weakly compact, $\mathbf{Q}$ κ-well-ordered, $\mathscr{P}_{<\kappa}(\mathbf{Q})$ is κ-well-ordered, we present subtle, almost ineffable (4.3, 4.4) and prove (4.5) that for κ almost ineffable, $\mathbf{Q}$ κ-well-ordered, $\mathscr{P}^1_{<\kappa}(\mathbf{Q})$ is κ-well-ordered. In the rest of the section we deal with the question when an X-barrier has a subbarrier of some depth; and (4.10) prove that a counterexample to "$\mathscr{P}_\alpha(\mathbf{Q})$ is D-well-ordered" can be chosen to have small hereditary cardinality.

§5: We introduce some classes of (labeled) trees $\mathcal{T}^l$, $\mathcal{T}^l(\mathbf{Q})$, $\mathcal{T}^l_{\leqq\alpha}(\mathbf{Q})$ (for $l = 0,1,2$) and embedding notions. The main theorem 5.3 says $\mathcal{T}^l(\mathbf{Q})$ is $[\kappa;\aleph_0]$-bqo if $\mathbf{Q}$ is ($l = 0,1,2$). For this we reduce it to the well-founded case in 5.4, and give a general criterion to $[\kappa;\lambda]$-X-bqo in 5.5. (Some previous proofs could have used it.) Theorem 5.6 suggests "$\mathbf{Q}\times(\omega,\leqq)$ is κ-X-bqo" as natural (e.g. equivalent to $\bigcup_\alpha \mathcal{P}^{**}_\alpha(\mathbf{Q})$ is κ-X-bqo) and in 5.7, 5.8 we compute the well-ordering number of $\mathcal{T}^0$, $\mathcal{T}^0(\mathbf{Q})$.

§6: We define $\mathfrak{M}_\lambda$ (union of λ scattered orders), $\mathfrak{M}_\lambda[\mathbf{Q}]$, and give their representation by trees (6.2–6.6) hence bounding the well-ordering number. We also get a bound for the well-ordering number of $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ (6.8, 6.9) and compute the well-ordering number of $\mathfrak{M}_\lambda$ (6.11, 6.12).

§7: We present $\mathcal{T}^{-2}$, $\mathcal{T}^{-1}$ (with which Nash-Williams has dealt) and other variants $\mathcal{T}^{-3}$, $\mathcal{T}^{-2.5}$, define local embeddability, and compute the well-ordering number of $\mathcal{T}^{-1}(\mathbf{Q})$, $\mathcal{T}^{-2}(\mathbf{Q})$, $\mathcal{T}^{-2.5}(\mathbf{Q})$, and $\mathcal{P}^1_{<\infty}(\mathbf{Q})$.

REMARK. Many questions remain open, but we have not tried to exhaust either the problems or the conclusions.

NOTATION. Let λ, μ, κ, χ be cardinals, usually infinite, α, β, γ, ξ, ζ, i, j be ordinals, δ a limit ordinal. Let $(-1)+\alpha$ be the unique γ, $\alpha = 1+\gamma$. Let l, m, n, r, k be natural numbers, η, ν, σ be sequences of ordinals, $l(\eta)$ the length of η (an ordinal), $\eta(i)$ the ith element of η, $\eta^\wedge\nu$ the concatanation of η and ν, and $\eta \trianglelefteq \nu$ ($\eta \triangleleft \nu$) means η is a (proper) initial segment of ν.

Let η^- be the unique sequence satisfying $\eta = \langle\eta(0)\rangle^\wedge\eta^-$.

Let $i\mathbf{R}^0_I j$ mean $i<j$ and $i\mathbf{R}^0_D j$ mean $i \neq j$.

Let X denote I or D, and $\eta\mathbf{R}^1_X\nu$ mean $\eta^- \trianglelefteq \nu$ and $\eta(0)\mathbf{R}^0_X\nu(0)$. When $\eta^- \trianglelefteq \nu$ let $\eta \cup^* \nu$ be $\langle\eta(0)\rangle^\wedge\nu$.

Let $|A|$ be the cardinality of A.

We call $\mathbf{Q} = (Q,\leqq)$ a quasi-order if $x \leqq x$, and $x \leqq y \wedge y \leqq z \Rightarrow x \leqq z$ for any $x,y,z \in Q$. We let $|\mathbf{Q}| = Q$ so $\|\mathbf{Q}\|$ is $|Q|$, called the cardinality of $\mathbf{Q}$, but sometimes we forget to distinguish between Q and $\mathbf{Q}$ (when the order is clear), so $a \in \mathbf{Q}$ means $a \in Q$. Note that we have not assumed $x \leqq y \wedge y \leqq x \Rightarrow x = y$, and let $x<y$ mean $x \leqq y$ but not $y \leqq x$.

Let $\mathrm{Seq}_\alpha(A) = \{\eta : \eta$ is a sequence from A of length $\alpha\}$,

$$\mathrm{Seq}_{<\alpha}(A) = \bigcup_{\beta<\alpha} \mathrm{Seq}_\beta(A).$$

For a set A of ordinals

$$X\,\mathrm{Seq}_\alpha(A)=\{\eta\in\mathrm{Seq}_\alpha(A):(\forall i)(i+1<l(\eta)\rightarrow\eta(i)\mathbf{R}^0_X\eta(i+1))\},$$

$$X\,\mathrm{Seq}_{<\alpha}(A)=\bigcup_{\beta<\alpha}X\,\mathrm{Seq}_\beta(A).$$

Note that any $\eta\in I\,\mathrm{Seq}_\alpha(A)$, for $\alpha\leqq\omega$, is increasing, and $\eta\in X\,\mathrm{Seq}_\alpha(A)$, $\beta<\alpha$ implies $\eta\restriction\beta\in X\,\mathrm{Seq}_\beta(A)$ (obviously Seq satisfies this too) and $\eta,\nu\in X\,\mathrm{Seq}_{<\omega}(A)$, $\eta\mathbf{R}^1_X\nu$ implies $\eta\cup^*\nu\in X\,\mathrm{Seq}_{<\omega}(A)$.

0.1. Definition. (1) If B is a set of finite sequences of ordinals, its domain $\mathrm{Dom}\,B=\mathrm{Dom}(B)$ is $\bigcup_{\eta\in B}\mathrm{Range}(\eta)$. We call $B\subseteq X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)$ an X-barrier if:

(a) for every $\eta\in X\,\mathrm{Seq}_\omega(\mathrm{Dom}\,B)$ for some $n<\omega$, $\eta\restriction n\in B$,

(b) no member of B is an initial segment of another,

(c) if $\eta\in B$, then there is no $\nu\lessdot\eta^-$ in B (but we do not forbid $\eta^-=\nu\in B$),

(d) B has at least two members (so the empty sequence $\langle\ \rangle$ is not in B) and $\mathrm{Dom}\,B$ has no last element when $X=I$ and $\mathrm{Dom}\,B$ is infinite when $X=D$.

(2) We define κ-I-barrier (κ-D-barrier) similarly, adding

(e) $\mathrm{Dom}\,B$ is a subset of κ of order type κ, but for notational simplicity we usually deal with the case $\mathrm{Dom}\,B=\kappa$.

Remark. In the definition of an I-barrier, we deviate from the definition Nash-Williams and Laver use, in (c).

Remark. A set B cannot be both an I-barrier and a D-barrier, except when

$$B=\{\langle\alpha\rangle:\alpha\in\mathrm{Dom}\,B\}$$

or

$$B=\{\langle\alpha\rangle:\alpha\in\mathrm{Dom}\,B-\{\alpha_0\}\}\cup\{\langle\alpha_0,\alpha\rangle:\alpha\in\mathrm{Dom}\,B-\{\alpha_0\}\},$$

where α_0 is the least ordinal in $\mathrm{Dom}\,B$.

0.2. Definition. Let $X\in\{I,D\}$, B an X-barrier, $\eta\in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)$.

(1) we define an ordinal $\alpha=\mathrm{Dp}(\eta,B)$ as follows:

(a) if there is $\nu\leqq\eta$, $\nu\in B$ then $\alpha=0$;

(b) if (a) fails then $\alpha=\bigcup\{\mathrm{Dp}(\eta^\frown\langle i\rangle,B)+1:\eta^\frown\langle i\rangle\in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)\}$.

(2) We define $\mathrm{Dp}(B)$, the depth of B, as $\mathrm{Dp}(\langle\ \rangle,B)$.

0.3. Claim. Definition 0.2 well defines $\mathrm{Dp}(\eta,B)$ uniquely, and it is $<|\mathrm{Dom}\,B|^+$.

Proof. By part (1)(b) of Definition 0.2, and (1)(a) of Definition 0.1.

0.4. Claim. Let B be an X-barrier, $X\in\{I,D\}$.

(1) If $\eta \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)$ then for some $\nu \in B$, η and ν are comparable, i.e., $\eta \leqq \nu$ or $\nu \leqq \eta$.

(2) Moreover, in (1), if $X = I$ and $A \subseteq \mathrm{Dom}\,B$ is unbounded, or $X = D$ and $A \subseteq \mathrm{Dom}\,B$ has $\geqq 2$ elements, we can assume $(\forall m)(l(\eta) \leqq m < l(\nu) \rightarrow \nu(m) \in A)$.

(3) If $\eta \in B$, then there is ν, $\eta \mathbf{R}^1_X \nu$, $\nu \in B$, and we can add the demands of (2).

(4) If $i \in \mathrm{Dom}\,B$, $i\mathbf{R}^0_X \eta(0)$, $\eta \in B$, then for some $k \leqq l(\eta)$, $\langle i \rangle^\frown(\eta \restriction k) \in B$.

PROOF. (1) and (2). We can find $\nu_0 \in X\,\mathrm{Seq}_\omega(\mathrm{Dom}\,B)$, $\eta \lessdot \nu_0$, such that $(\forall m)(l(\eta) \leqq m < \omega \rightarrow \nu_0(m) \in A)$; this is by part (d) of Definition 0.1(1).

Now by part (a) of the definition for some $k < \omega$, $\nu_0 \restriction k \in B$, and choose $\nu = \nu_0 \restriction k$.

(3) We act as in the proof of (2) for η^-, but if $l(\eta) = 1$, we require $\eta(0)\mathbf{R}^0_X \nu(0)$. The ν we get satisfies $\eta^- \leqq \nu$ as $\nu \lessdot \eta^-$ cannot hold by part (c) of Definition 0.1(1); and as $l(\eta) = 1$ implies $\eta(0)\mathbf{R}^0_X \nu(0)$, clearly $\eta\,\mathbf{R}^1_X \nu$.

(4) Easy, by (1) and Definition 0.1(1)(c) (remember $\langle\ \rangle \notin B$).

0.5. CLAIM. If B is an I-barrier, $\eta \in B$, $\nu \in I\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)$ and $\eta(i) < \nu(j)$ for every $i < l(\eta)$, $j < l(\nu)$, *then* we can find $\eta_l \in B$ $(l \leqq l(\eta))$ such that:

(a) $\eta_0 = \eta$,

(b) $\eta_l^- \leqq \eta_{l+1}$ for $l < l(\eta)$ and even $\eta_l \mathbf{R}^1_D \eta_{l+1}$,

(c) ν, $\eta_{l(\eta)}$ are $\leqq$-comparable, hence if $\nu \in B$ then $\nu = \eta_{l(\eta)}$.

(d) $\eta_l(k_1) = \eta_m(k_2)$ iff $k_1 + l = k_2 + m$ (where $0 \leqq k_1 < l(\eta_l)$, $0 \leqq k_2 < l(\eta_m)$).

PROOF. We define by induction on $l \leqq l(\eta)$, $\eta_l \in B$ such that:

(i) $\eta_0 = \eta$,

(ii) $\eta_{l-1}^- \leqq \eta_l$ and even $\eta_{l-1}\mathbf{R}^1_I \eta_l$,

(iii) η_l and $\langle \eta(m) : l \leqq m < l(\eta) \rangle^\frown \nu$ are comparable.

For $l = 0$ there is no problem. Suppose we have defined for l, let σ be the longer of the sequences η_l, $\langle \eta(m) : l \leqq m < l(\eta) \rangle^\frown \nu$. By 0.4(2), we let η_{l+1} be a sequence from B comparable with σ^-, hence 0.5(d) holds.

0.6. CLAIM. If B is a D-barrier, $\eta \in B$, $\nu \in D\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\,B)$ and $l(\nu) > 0 \rightarrow \eta(l(\eta) - 1) \neq \nu(0)$, *then* we can find $\eta_l \in B$ $(l \leqq l(\eta))$ such that:

(a) $\eta_0 = \eta$,

(b) $\eta_l^- \leqq \eta_{l+1}$ for $l < l(\eta)$ and even $\eta_l \mathbf{R}^1_D \eta_{l+1}$,

(c) ν, $\eta_{l(\eta)}$ are $\leqq$-comparable, hence if $\nu \in B$ then $\nu = \eta_{l(\eta)}$.

PROOF. Similar to that of Claim 0.5.

0.7. CLAIM. Suppose B is an X-barrier. Then for every $\eta, \nu \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)$, $\eta^- \leqq \nu$ implies

$$\mathrm{Dp}(\eta, B) \leqq \mathrm{Dp}(\eta^-, B) \geqq \mathrm{Dp}(\nu, B).$$

PROOF. By the definition of Dp (Definition 0.2) it is trivial that $\nu_1 \leqq \nu_2$ implies $\mathrm{Dp}(\nu_1, B) \geqq \mathrm{Dp}(\nu_2, B)$. So we have just to prove: $\mathrm{Dp}(\eta, B) \leqq \mathrm{Dp}(\eta^-, B)$, and we prove this by induction on $\mathrm{Dp}(\eta^-, B)$.

Case I. $\mathrm{Dp}(\eta^-, B) = 0$

By the definition of Dp, this implies that for some $k \leqq l(\eta^-)$, $(\eta^-) \restriction k \in B$. By Claim 0.4(4) this implies that for some $m \leqq k$, $\eta \restriction (m+1) = \langle \eta(0) \rangle ^\frown ((\eta^-) \restriction m) \in B$. Clearly $m+1 \leqq k+1 \leqq l(\eta^-)+1 \leqq l(\eta)$, so this implies $\mathrm{Dp}(\eta, B) = 0$.

Case II. $\mathrm{Dp}(\eta^-, B) > 0$

We can assume that $\mathrm{Dp}(\eta, B) > 0$ too (otherwise the conclusion is trivial).

Whenever $\eta ^\frown \langle i \rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)$ then $Dp((\eta ^\frown \langle i \rangle)^-, B) < \mathrm{Dp}(\eta^-, B)$ (as $\eta^- \lessdot (\eta ^\frown \langle i \rangle)^-$, by the definition of Dp), hence by the induction hypothesis

$$\mathrm{Dp}(\eta ^\frown \langle i \rangle, B) \leqq \mathrm{Dp}((\eta ^\frown \langle i \rangle)^-, B) = \mathrm{Dp}((\eta^-) ^\frown \langle i \rangle, B).$$

So by Definition 0.2(1)(b), the above inequality, and Definition 0.2(1)(b), respectively,

$$\begin{aligned} \mathrm{Dp}(\eta, B) &= \bigcup \{\mathrm{Dp}(\eta ^\frown \langle i \rangle, B) + 1 : \eta ^\frown \langle i \rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)\} \\ &\leqq \bigcup \{\mathrm{Dp}((\eta -) ^\frown \langle i \rangle, B) + 1 : (\eta -) ^\frown \langle i \rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)\} \\ &= \mathrm{Dp}(\eta^-, B). \end{aligned}$$

So we finish.

0.8. CLAIM. Suppose B is a κ-I-barrier, $\nu \in B \rightarrow l(\nu) \geqq 2$ and let us define $B^D = \{\eta \in D\,\mathrm{Seq}_{<\omega}(\kappa) : \eta$ is not monotonic, and for every $k < l(\eta)$, $\eta \restriction k$ is monotonic but not in $B\}$.[†]

Then $B^* = B \cup B^D$ is a κ-D-barrier (with the same domain κ).

PROOF. We check Definition 0.1. Clearly $\mathrm{Dom}\, B^* = \kappa$.

(a) If $\eta \in D\,\mathrm{Seq}_\omega(\kappa)$, if for some k, $\eta \restriction k \in B$, then $\eta \restriction k \in B^*$; if not, η cannot be monotonically increasing (as B is a $\kappa \cdot I$-barrier) nor monotonically

[†] The aim of this claim is to be used in 2.13 (with 0.9). If we put inside all decreasing sequences of a constant length, for any function F with small range, for some η, ν of this length, $\eta \mathbf{R}^1_D \nu$, $F(\eta) = F(\nu)$, which is what we want to avoid in 2.13.

decreasing (as κ is well-ordered). So there is a minimal k for which $\eta \restriction k$ is not monotonic; obviously $\eta \restriction k \in B^D \subseteq B^*$.

(b) Suppose $\eta, \nu \in B^*$ and $\eta \lessdot \nu$. It is impossible that $\eta, \nu \in B$ as B is a κ-I-barrier; also it is impossible that $\eta, \nu \in B^D$, or $\eta \in B$, $\nu \in B^D$ (by the definition of B^D). We are left with the case $\eta \in B^D$, $\nu \in B$, but ν is increasing (as $B \subseteq I\,\mathrm{Seq}_{<\omega}(\kappa)$) whereas η is not monotonic, contradiction.

(c) Suppose $\eta, \nu \in B^*$, $\nu \lessdot \eta^-$. Clearly $\eta, \nu \in B$ is impossible and also $\eta, \nu \in B^D$ (as then $\eta \restriction (l(\nu)+1)$ is not monotonic, $l(\nu)+1 < l(\eta)$). Obviously, $\nu \in B^D$, $\eta \in B$ is impossible (as η is monotonic, ν is not). At last, if $\eta \in B^D$, $\nu \in B$, so $l(\nu) \geqq 2$, then, as $\eta \restriction 3 \leqq \eta \restriction (l(\nu)+1) \lessdot \eta$ is monotonic, ν increasing, $\eta(0) < \eta(1) = \nu(0)$, so $\langle \eta(0) \rangle^\wedge \nu \in I\,\mathrm{Seq}_{<\omega}(\kappa)$. Hence for some $m \leqq l(\nu)$, $\langle \eta(0) \rangle^\wedge (\nu \restriction m) \in B$ so $\eta \restriction (m+1) \in B$ contradicts $\eta \in B^D$.

(d) $|B^*| \geqq |B| \geqq 2$.

0.9. Claim. Suppose B, B^D, B^* are as in 0.8,

$$B^D_{n,u} = \{\eta \in B^D : l(\eta) = n,\ \eta(0) < \eta(1)\},$$

$$B^D_{n,d} = \{\eta \in B^D : l(\eta) = n,\ \eta(0) > \eta(1)\},$$

then B^D is the disjoint union of the $B^D_{n,u}$, $B^D_{n,d}$ $(3 \leqq n < \omega)$ and if $\eta, \nu \in B^*$, $\eta \mathbf{R}^1_D \nu$ then exactly one of the following occurs:

(a) $\eta \in B$,

(b) $\eta \in B^D_{n+1,u}$, $\nu \in B^D_{n,u}$, $\nu = \eta^-$ for some $n \geqq 3$,

(c) $\eta \in B^D_{n+1,d}$, $\nu \in B^D_{n,d}$, $\nu = \eta^-$ for some $n \geqq 3$,

(d) $\eta \in B^D_{3,u}$, $\nu \in B^D_{n,d}$, $\eta^- = \nu \restriction 2$,

(e) $\eta \in B^D_{3,d}$, $\nu \in B^D_{n,u}$, $\eta^- = \nu \restriction 2$,

(f) $\eta \in B^D_{3,d}$, $\nu \in B$, $\eta^- = \nu \restriction 2$.

Proof. Easy.

0.10. Claim. Suppose B, B^D, B^* are as in 0.8. Then $\mathrm{Dp}(B^*) \leqq \mathrm{Max}\{\mathrm{Dp}(B), \kappa\}$.

Proof. We first establish, by induction on $\mathrm{Dp}(\eta, B^*)$, that for monotonically decreasing η, $1 < l(\eta) < \omega$, $\mathrm{Dp}(\eta, B^*) \leqq \eta(l(\eta)-1)+1$.

$\mathrm{Dp}(\eta, B^*) = \bigcup\{\mathrm{Dp}(\eta^\wedge\langle i \rangle, B^*)+1 : \eta^\wedge\langle i \rangle \in D\,\mathrm{Seq}_{<\omega}(\kappa)\}$. For $i > \eta(l(\eta)-1)$, $\mathrm{Dp}(\eta^\wedge\langle i \rangle, B^*) = 0$, while for $i < \eta(l(\eta)-1)$, $\mathrm{Dp}(\eta^\wedge\langle i \rangle, B^*) \leqq i+1$ by the induction hypothesis, so $\mathrm{Dp}(\eta, B)^* \leqq \eta(l(\eta)-1)+1$, as required.

Now we show, by induction on $\mathrm{Dp}(\nu, B^*)$, that for $\nu \in I\,\mathrm{Seq}_{<\omega}(\kappa)$, $1 < l(\nu)$, $\mathrm{Dp}(\nu, B^*) = \mathrm{Dp}(\nu, B)$. By the definitions $\mathrm{Dp}(\nu, B^*) = 0 \Leftrightarrow \mathrm{Dp}(\nu, B) = 0$. Assume now that $\mathrm{Dp}(\nu, B^*) > 0$, then $\mathrm{Dp}(\nu, B^*) = \bigcup\{\mathrm{Dp}(\nu\hat{\ }\langle i\rangle, B^*) + 1 : \nu\hat{\ }\langle i\rangle \in D\,\mathrm{Seq}_{<\omega}(\kappa)\}$. For $i < \nu(l(\nu) - 1)$, $\mathrm{Dp}(\nu\hat{\ }\langle i\rangle, B^*) = 0$, while for $i > \nu(l(\nu) - 1)$, $\mathrm{Dp}(\nu\hat{\ }\langle i\rangle, B^*) = \mathrm{Dp}(\nu\hat{\ }\langle i\rangle, B)$ by the induction hypothesis, so $\mathrm{Dp}(\nu, B^*) = \mathrm{Dp}(\nu, B)$.

If $l(\nu) = 1$, then it follows by what we have shown that $\mathrm{Dp}(\nu, B^*) \leqq \mathrm{Max}\{\mathrm{Dp}(\nu, B), \nu(0) + 1\}$. Hence $\mathrm{Dp}(\langle\ \rangle, B^*) \leqq \mathrm{Max}\{\mathrm{Dp}(\langle\ \rangle, B), \kappa\}$, as required.

§1. The basic generalization

Here we define the central notions generalizing well-ordering and better quasi-ordering, and generalize the basic theorem of Nash-Williams that better quasi-order is preserved by the operation $\mathscr{P}(\mathbf{Q})$.

As in Nash-Williams' works, we are interested in proving that various quasi-orders are κ-well-ordered, or κ-narrow. But we are naturally drawn to stronger notions (here, e.g., κ-X-bqo) as they are preserved by more operations. Our main conclusion is that $\mathbf{Q}$ is κ-X-bqo iff every $\mathscr{P}_\alpha(\mathbf{Q})$ is κ-X-well-ordered.

1.1. Definition. (1) A quasi-order $\mathbf{Q} = (Q, \leqq)$ is κ-well-ordered [κ-narrow] if for every $q_i \in Q$ $(i < \kappa)$ there are $i < j < \kappa$ [$i < \kappa$, $j < \kappa$, $i \neq j$] such that $q_i \leqq q_j$.

(2) A quasi-order $\mathbf{Q} = (Q, \leqq)$ is B-bqo (for B an X-barrier, $X \in \{I, D\}$, bqo standing for better quasi-order) *if* for every $q_\eta \in Q$ $(\eta \in B)$ we can find $\eta, \nu \in B$ such that $\eta \mathbf{R}^1_X \nu$ and $q_\eta \leqq q_\nu$. For emphasis we write "B-X-bqo".

(3) A quasi-order $\mathbf{Q}$ is (κ, α)-X-bqo if for every κ-X-barrier B of depth $\leqq \alpha$, $\mathbf{Q}$ is B-bqo.

(4) A quasi-order $\mathbf{Q}$ is κ-X-bqo if it is (κ, α)-X-bqo for every α (in fact, for every $\alpha < \kappa^+$).

(5) Let κ-I-well-order mean κ-well-order, and κ-D-well-order mean κ-narrow.

We do not list the obvious implications and monotonicity properties.

Now we define some simple operations on quasi-orders.

1.2. Definition. (1) For quasi-orders $\mathbf{Q}_1 = (Q_1, \leqq)$, $\mathbf{Q}_2 = (Q_2, \leqq)$ we define their product; its set of elements is $Q_1 \times Q_2 = \{(q_1, q_2) : q_1 \in Q_1, q_2 \in Q_2\}$ and for $q_1, q_1' \in Q_1$, $q_2, q_2' \in Q_2$

$$(q_1, q_2) \leqq (q_1', q_2') \qquad \text{iff} \quad q_1 \leqq q_1' \text{ and } q_2 \leqq q_2'.$$

(2) $\Pi_{i<\alpha} \mathbf{Q}_i$ is defined similarly, and $\mathbf{Q}^\alpha$ is $\Pi_{i<\alpha} \mathbf{Q}_i$ where $\mathbf{Q}_i = \mathbf{Q}$.

1.3. Definition. (1) $\mathcal{P}(Q)$ is the family of subsets of Q, and if $\mathbf{Q} = (Q, \leqq)$, we let $\leqq$ be the following order on it:

$A \leqq B$ iff there is a function $h : A \to B$, such that $q \leqq h(q)$ for every $q \in A$.

So we let $\mathcal{P}(\mathbf{Q}) = (\mathcal{P}(|\mathbf{Q}|), \leqq)$.

(2) $\mathcal{P}_{<\kappa}(A) = \{C \subseteq A : |C| < \kappa\}$, and $\mathcal{P}_{<\kappa}(\mathbf{Q})$ is defined naturally.

(3) For any set A we define $\mathcal{P}_\alpha(A)$ (α an ordinal) by induction on α as follows:

$$\mathcal{P}_0(A) = A,$$

$$\mathcal{P}_{\alpha+1}(A) = \mathcal{P}_\alpha(A) \cup \mathcal{P}(\mathcal{P}_\alpha(A)),$$

$$\mathcal{P}_\delta(A) = \bigcup_{\alpha<\delta} \mathcal{P}_\alpha(A) \qquad \text{(for } \delta \text{ limit)},$$

$$\mathcal{P}_{<\infty}(A) = \bigcup_{\alpha} \mathcal{P}_\alpha(A).$$

We treat here the elements of A as *urelements.* Clearly $\mathcal{P}_\alpha(A)$ increases with α, and for $x \in \mathcal{P}_\alpha(A)$ let $T_c(x)$, its transitive closure, be the minimum transitive set which $\supseteq \{x\}$.

(4) For any quasi-order $\mathbf{Q} = (Q, \leqq)$ and ordinal α, we define the quasi-order $\mathcal{P}_\alpha(\mathbf{Q}) = (\mathcal{P}_\alpha(Q), \leqq)$ by induction on α:

For $\alpha = 0$, $\alpha = \delta$ there is no problem; for $\alpha = \beta + 1$, $A_1 \leqq A_2$ *iff* (a) $A_1, A_2 \in \mathcal{P}_\beta(A)$, $A_1 \leqq A_2$, or (b) $A_1, A_2 \in \mathcal{P}_\alpha(Q)$ but not $\{A_1, A_2\} \subseteq \mathcal{P}_\beta(Q)$, and

(i) there is a function $f : A_1 \to A_2$, $(\forall q \in A_1)\ q \leqq f(q)$, or

(ii) $A_2 = q \in Q$ and $(\forall t \in Tc(A_1))\ (t \in Q \to t \leqq q)$, or

(iii) $A_1 \leqq A' \in A_2$ for some A' (so $A_1 \in \mathcal{P}_\beta(Q)$).

(5) $\mathcal{P}^*_\alpha(\mathbf{Q}) = (\mathcal{P}_\alpha(Q), \leqq^*)$ is defined similarly except that we omit 1.3(4)(b)(ii) and demand in 1.3(4)(b)(i) $A_2 \notin \mathcal{P}_\beta(\mathbf{Q})$ or $A_1 \in \mathcal{P}_\beta(\mathbf{Q})$.

(6) $\mathcal{P}^{**}_\alpha(\mathbf{Q})$ is defined similarly to $\mathcal{P}^*_\alpha(\mathbf{Q})$ but we omit the empty sets.

(7) $\mathcal{P}^0_\alpha(\mathbf{Q})$ is defined similarly to $\mathcal{P}_\alpha(\mathbf{Q})$, but we omit the empty sets.

Some obvious facts are

1.4. Claim. Suppose κ is a weakly compact cardinal.

(1) If $\mathbf{Q}_1$, $\mathbf{Q}_2$ are κ-well-ordered *then* $\mathbf{Q}_1 \times \mathbf{Q}_2$ is κ-well-ordered.

(2) If $\alpha < \kappa$, $\mathbf{Q}_i$ $(i < \alpha)$ are κ-well-ordered then $\Pi_{i<\alpha} \mathbf{Q}_i$ is κ-well-ordered.

1.5. Claim. (1) A quasi-order $\mathbf{Q}$ is κ-well-ordered iff it is $(\kappa, 1)$-I-bqo;

(2) A quasi-order $\mathbf{Q}$ is κ-narrow iff it is $(\kappa, 1)$-D-bqo.

(3) If $\kappa \leqq \kappa'$, $\alpha \geqq \alpha'$ then any (κ, α)-X-bqo quasi-order is (κ', α')-X-bqo.

1.6. CLAIM. If $\mathbf{Q}^\lambda$ is κ-well-ordered [κ-narrow] *then* $\mathcal{P}_{<\lambda^+}(\mathbf{Q})$ is κ-well-ordered [κ-narrow].

1.7. CLAIM. Suppose $A_1, A_2 \in \mathcal{P}_\alpha(\mathbf{Q})$. Then $A_1 \leqq A_2$ *iff* one of the following holds:

(i) $A_1, A_2 \notin \mathbf{Q}$, and for every $a_1 \in A_1$ there is $a_2 \in A_2$, such that $a_1 \leqq a_2$;

(ii) $A_2 \in \mathbf{Q}$, $A_1 \notin \mathbf{Q}$ and $(\forall a_1 \in A_1)$ $a_1 \leqq A_2$ (equivalently, $[\forall q \in (Tc(A_1) \cap \mathbf{Q})]$ $q \leqq A_2$);

(iii) $A_1 \in \mathbf{Q}$, $A_2 \notin \mathbf{Q}$, and $(\exists a_2 \in A_2)$ $A_1 \leqq a_2$ (equivalently, $[\exists q \in (Tc(A_2) \cap \mathbf{Q})]$ $A_1 \leqq q$);

(iv) $A_1, A_2 \in \mathbf{Q}$, $A_1 \leqq A_2$.

PROOF. The proof proceeds by induction on α, following Definition 1.3(4); the details are left to the reader.

The main result of this section is

1.8. THEOREM. *Let* $X \in \{I, D\}$. *If* $\mathbf{Q}$ *is* κ-X-bqo, *then so is* $\mathcal{P}_\alpha(\mathbf{Q})$ *(for any* α*).*

REMARK. If $\mathbf{Q}$ is a $(\kappa, \gamma(\alpha, \beta))$-$X$-bqo, then $\mathcal{P}_\beta(\mathbf{Q})$ is (κ, α)-X-bqo for some $\gamma(\alpha, \beta)$ which can be computed (but we have not computed it).

To clarify the proof, we proceed first to prove a similar fact about $\mathcal{P}(\mathbf{Q})$. We start with a claim on barriers:

1.9. CLAIM. (1) Let $X \in \{I, D\}$, B an X-barrier, and let

$$B' = \{\eta \cup^* \nu : \eta \in B,\ \nu \in B,\ \eta \mathbf{R}^1_X \nu\}.$$

Then B' is a barrier, $\operatorname{Dom} B' = \operatorname{Dom} B$, and $\operatorname{Dp}(B') \leqq \operatorname{Dp}(B) + 1$.

(2) Let B be an X-barrier, $X \in \{I, D\}$, $C \subseteq B$ and

$$B'_C = C \cup \{\eta \cup^* \nu : \eta \in B - C,\ \nu \in B,\ \eta \mathbf{R}^1_X \nu\}.$$

Then B'_C is a barrier with domain $\operatorname{Dom} B$ and depth $\leqq \operatorname{Dp}(B) + 1$.

PROOF. (1) Let us check the definition of an X-barrier (Definition 0.1(1)):

(a) We shall show that for every $\eta \in X\operatorname{Seq}_\omega(\operatorname{Dom} B)$, for some n, $\eta \restriction n \in B'$ (thus establishing $\operatorname{Dom} B \subseteq \operatorname{Dom} B'$).

As B is an X-barrier and $\eta \in X\operatorname{Seq}_\omega(\operatorname{Dom} B)$ for some m, $\eta \restriction m \in B$. Also $\eta^- \in X\operatorname{Seq}_\omega(\operatorname{Dom} B)$ hence for some k, $(\eta^-) \restriction k \in B$. Clearly by 0.1(1)(c), $(\eta \restriction m)\mathbf{R}^1_X(\eta^- \restriction k)$ and $\eta \restriction \max\{m, k+1\} = (\eta \restriction m) \cup^* (\eta^- \restriction k) \in B'$.

(b) No member of B' is an initial segment of another.

Suppose $\sigma_l = \eta_l \cup^* \nu_l$, $\eta_l \in B$, $\nu_l \in B$, $\eta_l \mathbf{R}^1_X \nu_l$, for $l = 0, 1$ and $\sigma_1 \lessdot \sigma_2$, and we shall get a contradiction.

As $\sigma_1 \not\lessdot \sigma_2$ clearly $\sigma_1^- \not\lessdot \sigma_2^-$. But $\nu_l = \sigma_l^-$, so $\nu_0 \lessdot \nu_1$, but this contradicts $\nu_0, \nu_1 \in B$.

(c) Similar to (b).

(d) B' has at least two members. Obvious by (a) here.

Now we check $\mathrm{Dom}\, B' = \mathrm{Dom}\, B$. One inclusion follows by (a) here, the other by the definition of B'. So $\mathrm{Dom}\, B$ satisfies the requirement in 0.1(1)(d).

We are left with $\mathrm{Dp}(B') \leqq \mathrm{Dp}(B) + 1$. To prove this we prove by induction on $\alpha \leqq \mathrm{Dp}(B)$ that:

(*) if $\eta \neq \langle\ \rangle$, $\eta \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)$ and $\mathrm{Dp}(\eta^-, B) \leqq \alpha$, then $\mathrm{Dp}(\eta, B') \leqq \alpha$.

First let $\alpha = 0$, so as $\mathrm{Dp}(\eta, B) \leqq \alpha$ (by 0.7) for some k, $\eta \restriction k \in B$, and as $\mathrm{Dp}(\eta^-, B) \leqq \alpha$ for some m, $(\eta^- \restriction m) \in B$. So $(\eta \restriction k)^- \trianglelefteq (\eta^- \restriction m)$, and $(\eta \restriction k) \cup^* (\eta^- \restriction m)$ belongs to B' and is an initial segment of η, hence $\mathrm{Dp}(\eta, B') = 0$.

Second, let $\alpha > 0$, so if $\eta^\wedge\langle i\rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)$ then $\mathrm{Dp}((\eta^\wedge\langle i\rangle)^-, B) < \mathrm{Dp}(\eta^-, B) \leqq \alpha$ hence by the induction hypothesis:

$$\mathrm{Dp}(\eta^\wedge\langle i\rangle, B') \leqq \mathrm{Dp}((\eta^\wedge\langle i\rangle)^-, B).$$

So

$$\begin{aligned}\mathrm{Dp}(\eta, B') &= \bigcup\{\mathrm{Dp}(\eta^\wedge\langle i\rangle, B') + 1 : \eta^\wedge\langle i\rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)\}\\ &\leqq \bigcup\{\mathrm{Dp}((\eta^\wedge\langle i\rangle)^-, B) + 1 : \eta^\wedge\langle i\rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)\}\\ &= \bigcup\{\mathrm{Dp}((\eta^-)^\wedge\langle i\rangle, B) + 1 : (\eta^-)^\wedge\langle i\rangle \in X\,\mathrm{Seq}_{<\omega}(\mathrm{Dom}\, B)\}\\ &= \mathrm{Dp}(\eta^-, B).\end{aligned}$$

So we proved (*), so

$$\begin{aligned}\mathrm{Dp}(B') = \mathrm{Dp}(\langle\ \rangle, B') &= \bigcup\{\mathrm{Dp}(\langle i\rangle, B') + 1 : i \in \mathrm{Dom}\, B = \mathrm{Dom}\, B'\}\\ &\leqq \bigcup\{\mathrm{Dp}(\langle i\rangle^-, B) + 1 : i \in \mathrm{Dom}\, B\}\\ &= \bigcup\{\mathrm{Dp}(\langle\ \rangle, B) + 1 : i \in \mathrm{Dom}\, B\}\\ &= \mathrm{Dp}(\langle\ \rangle, B) + 1 = \mathrm{Dp}(B) + 1.\end{aligned}$$

(2) Similar proof.

1.10. Theorem. *Let B be an X-barrier, $\mathbf{Q}$ a quasi-order. If $\mathbf{Q}$ is B'-X-*bqo, *then $\mathcal{P}(\mathbf{Q})$ is B-X-*bqo.

Proof. For proving $\mathcal{P}(\mathbf{Q})$ is B-X-bqo it suffices to prove:

(*) if $A_\eta \in \mathcal{P}(\mathbf{Q})$ for $\eta \in B$ then for some $\eta \in B$, $\nu \in B$, $\eta \mathbf{R}_X^1 \nu$ and $A_\eta \leqq A_\nu$.

Suppose (*) fails, and $A_\eta\ (\eta \in B)$ exemplify this failure.

For every $\sigma \in B'$, there are $\eta \in B$, $\nu \in B$ such that $\eta \mathbf{R}^1_X \nu$ and $\sigma = \eta \cup^* \nu$, and the pair $\langle \eta, \nu \rangle$ is unique (for σ). Choose $q_\sigma \in A_\eta$ such that for no $q \in A_\nu$ does $q_\sigma \leqq q$; this is possible as not $A_\eta \leqq A_\nu$. We shall prove that $\langle q_\sigma : \sigma \in B' \rangle$ exemplify Q is not B'-X-bqo, getting a contradiction to the assumption, thus finishing.

So suppose $\sigma_0 \mathbf{R}^1_X \sigma_1$, and $\sigma_0, \sigma_1 \in B'$, and we have to prove that not $q_{\sigma_0} \leqq q_{\sigma_1}$. For $l = 0, 1$ there are $\eta_l, \nu_l \in B$, $\eta_l \mathbf{R}^1_X \nu_l$ such that $\sigma_l = \eta_l \cup^* \nu_l$. Clearly $\nu_0 = \sigma_0^- \leqq \sigma_1$ and $\eta_1 \leqq \sigma_1$, hence ν_0, η_1 are comparable; but both are in B, so by Definition 0.1(1)(b), $\nu_0 = \eta_1$.

Now $q_{\sigma_1} \in A_{\eta_1}$ (by the choice of q_{σ_1}) hence $q_{\sigma_1} \in A_{\nu_0}$, but $(\forall q \in A_{\nu_0})(q_{\sigma_0} \not\leqq q)$ so not $q_{\sigma_0} \leqq q_{\sigma_1}$, as required.

Now we return to

Proof of Theorem 1.8. Let B be a κ-X-barrier; we have to prove that $\mathcal{P}_\alpha(\mathbf{Q})$ is B-X-bqo. Suppose $A_\eta \in \mathcal{P}_\alpha(\mathbf{Q})$ $(\eta \in B)$ exemplify $\mathcal{P}_\alpha(\mathbf{Q})$ is not B-X-bqo. We shall prove that $\mathbf{Q}$ is not κ-X-bqo, thus finishing.

We now define by induction on n, B_n and $t^n_\eta \in \mathcal{P}_\alpha(\mathbf{Q})$ (for every $\eta \in B_n$) as follows:

For $n = 0$, $B_n = B$, and

$$t^n_\eta = A_\eta \qquad \text{for every } \eta \in B_n.$$

For $n+1$. We let $B_{n+1} = \{\eta \cup^* \nu : \eta \in B_n,\ \nu \in B_n,\ \eta \mathbf{R}^1_X \nu, \text{ and } t^n_\eta \notin \mathbf{Q}\} \cup \{\eta : \eta \in B_n,\ t^n_\eta \in \mathbf{Q}\}$.

We shall prove that $\eta \mathbf{R}^1_X \nu$, $\eta \in B_n$, $\nu \in B_n$ implies "not $t^n_\eta \leqq t^n_\nu$". (See Fact A below.) So for $\sigma = \eta \cup^* \nu \in B_{n+1}$ as above choose $t^{n+1}_\sigma \in t^n_\eta$ such that for no $t' \in t^n_\nu$, $t^{n+1}_\sigma \leqq t'$ and if $t^n_\nu \in \mathbf{Q}$, then not $t^{n+1}_\sigma \leqq t^n_\nu$ (possible by 1.7). If $\eta \in B_n$, $t^n_\eta \in \mathbf{Q}$ then $t^{n+1}_\eta = t^n_\eta$. We let $t^n(\eta) = t^n_\eta$.

Let $B^* = \{\eta$: for some n, $\eta \in B_n$, and $t^n_\eta \in \mathbf{Q}\}$. We shall prove that B^* is an X-barrier with domain $\operatorname{Dom} B$ and if $\eta \in B_n$ and $t^n_\eta \in \mathbf{Q}$ then $(\forall m \geqq n)$ $(\eta \in B_m \wedge t^m_\eta = t^n_\eta)$. Let t_η be t^n_η; $\eta, \nu \in B^*$, $\eta \mathbf{R}^1_X \nu$ implies "not $t_\eta \leqq t_\nu$", thus proving $\mathbf{Q}$ is not B^*-X-bqo, hence not κ-X-bqo.

Fact A. B_n is an X-barrier, $\operatorname{Dom} B_n = \operatorname{Dom} B$; and not $t^n_\eta \leqq t^n_\nu$ whenever $\eta \mathbf{R}^1_X \nu$, $\eta \in B_n$, $\nu \in B_n$.

We prove this fact by induction on n; for $n = 0$ it is an assumption so suppose it holds for n and we shall prove it for $n+1$. The first part follows by 1.9(2); the proof of the second part is similar to that of 1.10 and is left to the reader.

FACT B. If $\eta \in B_n$ and $t^n_\eta \in \mathbf{Q}$, then $\eta \in B_m$, $t^m_\eta = t^n_\eta$ for $m \geqq n$.

We prove this by induction on $m \geqq n$. The induction step is by the definition of B_{n+1}.

CONVENTION C. For $\eta \in B^*$, t_η is t^n_η for every n such that $\eta \in B_n$, $t^n_\eta \in \mathbf{Q}$.

FACT D. If $\eta \in X\,\mathrm{Seq}_\omega(\mathrm{Dom}\,B)$, then for some $m, \eta \restriction n \in B^*$.

As B is a κ-X-barrier, there are η_l $(l < \omega)$ such that $\eta_l \in B$ and $\eta_l \lessdot \langle \eta(l+m) : m < \omega \rangle$.

We now define by induction on $k < \omega$, $\alpha(k) \leqq \omega$ and η^k_l $(l < \alpha(k))$ such that:

(a) $\alpha(0)$ is defined, $\alpha(k+1) \leqq \alpha(k)$, $\alpha(k) > 0$ and $\alpha(k+1)$ is defined iff $\alpha(k) > 1$,

(b) for every $l < \alpha(k)$, $\eta^k_l \in B_k$; $\eta^0_l = \eta_l$,

(c) if $l+1 < \alpha(k)$ then $t^k(\eta^k_l) \notin \mathbf{Q}$ and $(\eta^k_l)\mathbf{R}^1_X(\eta^k_{l+1})$,

(d) if $\alpha(k) < \omega$, $l+1 = \alpha(k)$ then $t^k(\eta^k_l) \in \mathbf{Q}$,

(e) if $\alpha(k+1)$ is defined then $t^{k+1}(\eta^{k+1}_0) \in t^k(\eta^k_0)$,

(f) $\eta^k_l \lessdot \langle \eta(l), \eta(l+1), \eta(l+2), \cdots \rangle$.

This is sufficient, because by (e), as $\in$ is well-founded, necessarily for some k, $\alpha(k+1)$ is not defined but $\alpha(k)$ is defined (remember that by (a) $\alpha(0)$ is defined). So by (a), $\alpha(k) = 1$, hence by (d), $t^k(\eta^k_0) \in \mathbf{Q}$, but $\eta^k_0 \in B_k$, so $\eta^k_0 \in B^*$. As by (f) $\eta^k_0 \lessdot \eta$ we shall finish the proof of Fact D.

So we have only to carry the induction. For $k = 0$, $\alpha(k)$ is the first l such that $l \geqq 1$, $t^0(\eta^0_{l-1}) \in \mathbf{Q}$ if there exists such l, and ω otherwise. For $k+1$ define $\eta^{k+1}_l = \eta^k_l \cup^* \eta^k_{l+1}$ for $l < \alpha(k) - 1$, and $\eta^{k+1}_l = \eta^k_l$ for $l = \alpha(k) - 1$ (if $\alpha(k) < \omega$) and then define $\alpha(k+1)$ (if $\alpha(k) > 1$) to satisfy (c) and (d).

FACT E. B^* is an X-barrier, $\mathrm{Dom}\,B^* = \mathrm{Dom}\,B$, and not $t_\eta \leqq t_\nu$ when $\eta \mathbf{R}^1_X \nu$, $\eta \in B^*$, $\nu \in B^*$.

Just sum up the previous facts; most properties can be reduced to properties of B_n by Fact B, which then hold by Fact A; the rest follows by Fact D.

As we have indicated before, we use now Fact E to obtain a contradiction to the assumption of Theorem 1.8.

1.11. THEOREM. *The quasi-order* $\mathbf{Q}$ *is* κ-X-bqo iff *for every* α

$\mathcal{P}_\alpha(\mathbf{Q})$ *is* κ-X-*well-ordered* iff *for every* $\alpha < \kappa^+$,

$\mathcal{P}_\alpha(\mathbf{Q})$ *is* κ-X-*well-ordered*, iff $\mathcal{P}_\gamma(\mathbf{Q})$ *is* κ-X-bqo (*for any specific* γ).

PROOF. The fourth phrase follows from the first by 1.8 and implies it as we can embed $\mathbf{Q}$ into $\mathcal{P}_\gamma(\mathbf{Q})$.

The first phrase implies the second by Theorem 1.8, the second phrase implies the third trivially. The third phrase implies the first by the following lemma, thus finishing the proof.

1.12. Lemma. (1) *Suppose* $\mathbf{Q}$ *is not* (κ,α)-X-bqo, *then* $\mathcal{P}_{(-1)+\alpha}(\mathbf{Q})$ *is not* κ-X-*well-ordered.*

(2) *Suppose*

(A) $C \subseteq X\,\mathrm{Seq}_{<\omega}(\kappa)$, *C closed under initial segments,* $f: C \to \kappa^+$ *is such that* $f(\eta) = \bigcup\{f(\eta^\wedge\langle i\rangle)+1 : \eta^\wedge\langle i\rangle \in C\}$ *and* $|D| = \kappa$, *where* $D = \{i : \langle i\rangle \in C\}$, *and let* $C_0 = \{\eta \in C : f(\eta) = 0\}$;

(B) $q_\eta \in \mathbf{Q}$ *for* $\eta \in C_0$; *and* $\eta, \nu \in C_0$, $\eta \mathbf{R}^1_X \nu$ *implies not* $q_\eta \leqq q_\nu$;

(C) *for every* $\eta \in C - C_0$, $\nu \in C$ *if* $\eta \mathbf{R}^1_X \nu$, $l(\eta) = l(\nu)$ *then* $\eta \cup^* \nu \in C$;

(D) *there are no* $\eta \in C_0$, $\nu \in C_0$, $\nu \lessdot \eta$.

Then

$$\mathcal{P}_{(-1)+f(\langle\ \rangle)}(\mathbf{Q}) \text{ is not } \kappa\text{-}X\text{-well-ordered.}$$

Proof. (1) We will show that part (1) follows easily from part (2).

As $\mathbf{Q}$ is not (κ,α)-X-bqo, there is a κ-X-barrier B, of depth $\leqq \alpha$, and q_η $(\eta \in B)$ such that $\eta \mathbf{R}^1_X \nu$ implies not $q_\eta \leqq q_\nu$. Now let

$$C = \{\eta \restriction k : \eta \in B,\ k \leqq l(\eta)\};$$

$$f: C \to \kappa^+ \text{ is defined by } f(\eta) = \mathrm{Dp}(\eta, B).$$

Clearly $C_0 = B$ and all the assumptions of 1.12(2) hold (part C) by 0.4(1), and $f(\langle\ \rangle) = \mathrm{Dp}(\langle\ \rangle, B) = \mathrm{Dp}(B) \leqq \alpha$. So by 1.12(2), $\mathcal{P}_{(-1)+f(\langle\ \rangle)}(\mathbf{Q})$ is not κ-X-well-ordered. As $f(\langle\ \rangle) \leqq \alpha$, $\mathcal{P}_{(-1)+f(\langle\ \rangle)}(\mathbf{Q}) \subseteq \mathcal{P}_{(-1)+\alpha}(\mathbf{Q})$ hence $\mathcal{P}_{(-1)+\alpha}(\mathbf{Q})$ is not κ-X-well-ordered.

(2) Let $g(\eta) = (-1) + (f(\eta)+1)$ (so $f(\eta) = 0 \Leftrightarrow g(\eta) = 0$). We define $t_\eta \in \mathcal{P}_{g(\eta)}(\mathbf{Q})$ for $\eta \in C$, by induction on $f(\eta)$. If $f(\eta) = 0$ then $\eta \in C_0$, and let $t_\eta = q_\eta \in \mathbf{Q} = \mathcal{P}_0(\mathbf{Q})$ (by Definition 1.3). If $f(\eta) > 0$, let $t_\eta = \{t_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in C\}$. Note that $\eta^\wedge\langle i\rangle \in C$ implies $f(\eta) > f(\eta^\wedge\langle i\rangle)$ which implies $t_{\eta^\wedge\langle i\rangle}$ is already defined and belongs to $\mathcal{P}_{g(\eta^\wedge\langle i\rangle)}(\mathbf{Q}) \subseteq \mathcal{P}_{(-1)+f(\eta)}(\mathbf{Q})$. Hence $t_\eta \subseteq \mathcal{P}_{(-1)+f(\eta)}(\mathbf{Q})$, so $t_\eta \in \mathcal{P}_{g(\eta)}(\mathbf{Q})$.

Now for $i \in D$, $g(\langle i\rangle) = (-1) + (f(\langle i\rangle)+1) \leqq (-1) + f(\langle\ \rangle)$ hence $t_{\langle i\rangle} \in \mathcal{P}_{g(\langle i\rangle)}(\mathbf{Q}) \subseteq \mathcal{P}_{(-1)+f(\langle\ \rangle)}(\mathbf{Q})$. As $|D| = \kappa$, $D \subseteq \kappa$, it suffices to prove that:

$$(*) \qquad i, j \in D,\ i \mathbf{R}^0_X j \text{ implies not } t_{\langle i\rangle} \leqq t_{\langle j\rangle}.$$

So suppose i, j form a counterexample to $(*)$. We now define by induction on l, ordinals $i(l)$, $j(l)$ such that:

(a) $i(0)=i,\ j(0)=j$,

(b) $\eta_l=\langle i(0),\cdots,i(l)\rangle\in C,\ \nu_l=\langle j(0),\cdots,j(l)\rangle\in C$,

(c) $i(l+1)$ is defined iff $f(\eta_l)>0$, and then it is $j(l)$,

(d) $t_{\eta_l}\leqq t_{\nu_l}$.

For $l=0$, use (a). If we have defined $i(0),j(0),\cdots,i(l),j(l)$ and $f(\eta_l)>0$, then let $i(l+1)=j(l)$. Now $\eta_{l+1}\in C$ by assumption (C) since $i(0)\mathbf{R}^0_x j(0)$.

By the definition of t_{η_l}, $t_{\eta_{l+1}}\in t_{\eta_l}$. Hence by 1.7 for some $t\in t_{\nu_l}$, $t_{\eta_{l+1}}\leqq t$ or $t_{\nu_l}\in\mathbf{Q}$. In the first case by the definition of t_{ν_l}, for some j_{l+1}, $\nu_l\hat{}\langle j_{l+1}\rangle\in C$ and $t=t_{\nu\hat{}\langle j_{l+1}\rangle}$. So we carry the induction. In the second case, necessarily $\nu_l\in C_0$ (by t_{ν_l}'s definition; recall that we treat the members of $\mathbf{Q}$ as urelements).

By induction on $f(\eta)$ one can easily show that for each $\eta\in C$ there is $\eta^*\in C_0$, $\eta\leqq\eta^*$. Thus, let $\eta^*\in C_0$, $\eta_{l+1}\leqq\eta^*$. Then $\nu_l=(\eta_{l+1})^-\leqq(\eta^*)^-$, so by assumption (D), $\nu_l=(\eta^*)^-$, hence $\eta_{l+1}=\eta^*\in C_0$.

By 1.7, $t_{\eta_{l+1}}\leqq t_{\nu_l}$, and clearly $\eta_{l+1}\mathbf{R}^1_X\nu_l$, contradicting assumption (B); so the second case never occurs, and we can carry the induction.

As $\eta_l\lessdot\eta_{l+1}$, clearly $f(\eta_l)>f(\eta_{l+1})$, hence for some $m, f(\eta_m)=0$; so $i(m+1)$ is not defined. So $t_{\eta_m}\in\mathbf{Q}$, $\eta_m\in C_0$. Now we can define by induction on $l\geqq m$, $j(l)$, such that $\nu_l=\langle j(0),\cdots,j(l)\rangle\in C$, $j(l+1)$ is defined iff $f(\nu_l)>0$, and $t_{\eta_m}\leqq t_{\nu_l}$ (for $l=m$, $j(l)$ is already defined). Again we can carry the induction and for some n, ν_n is defined and is in C_0. So $t_{\nu_n}\in\mathbf{Q}$, $t_{\eta_m}\leqq t_{\nu_n}$, $\eta_m\mathbf{R}^1_X\nu_n$ and again we get a contradiction to assumption (B).

§2. Existence theorem and a stronger notion suitable for powers

The interest in the theorems of §1 is not clear till we find non-trivial examples of κ-X-bqo (in addition to the $\aleph_0$-I-bqo with which Nash-Williams dealt). It is also not clear what the additional case $X=D$ gives us. Another fault is that we do not have any parallel of the fact "if $\mathbf{Q}$ is bqo, then so is $\mathbf{Q}^2=\mathbf{Q}\times\mathbf{Q}$". This section suggests remedies.

2.1. Definition. (1) $\lambda\rightarrow(\mu)^{<\omega}_\kappa$ if for every function F from $I\,\mathrm{Seq}_{<\omega}(\lambda)$ to κ, there is a set $A\subseteq\lambda$ of cardinality μ such that for every $n<\omega$, $F\restriction(I\,\mathrm{Seq}_n(A))$ is constant (this relation has obvious monotonicity properties). We define $\lambda\rightarrow(\mu)^n_\kappa$ similarly.

(2) We call λ a *Ramsey* cardinal iff $\lambda\rightarrow(\lambda)^{<\omega}_\kappa$ for every $\kappa<\lambda$.

(3) We call λ a weakly compact cardinal iff $\lambda\rightarrow(\lambda)^2_2$ and $\lambda>\aleph_0$.

2.2. Theorem. (1) *Every Ramsey cardinal is weakly compact, and every weakly compact is strongly inaccessible, and λ is a Ramsey cardinal iff $\lambda\rightarrow(\lambda)^{<\omega}_2$, and $\lambda>\aleph_0$ is weakly compact iff $\lambda\rightarrow(\lambda)^n_\mu$ for every $n<\omega$, $\mu<\lambda$.*

(2) *If κ is Ramsey, B a κ-I-barrier, then for some $A \subseteq \operatorname{Dom} B$, $|A| = \kappa$ and $B \cap (I\operatorname{Seq}_{<\omega}(A)) = I\operatorname{Seq}_n(A)$ for some n. For weakly compact cardinals this holds for every I-barrier of depth $< \omega$.*

PROOF. For part (1) see [5], and part (2) is trivial.

REMARK. See 4.7.

2.3. DEFINITION. (1) $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ means that for every function F from $I\operatorname{Seq}_{<\omega}(\kappa)$ to χ, there are $\alpha(0) < \alpha(1) < \cdots < \alpha(n) < \cdots$ such that for every n, $F(\langle \alpha(0), \cdots, \alpha(n-1)\rangle) = F(\langle \alpha(1), \cdots, \alpha(n)\rangle)$. Notice that if we replace $I\operatorname{Seq}_{<\omega}(\kappa)$ by $D\operatorname{Seq}_{<\omega}(\kappa)$ we obtain an equivalent definition.

(2) We call κ *beautiful* if $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ for every $\chi < \kappa$ or $\kappa = \aleph_0$. We call κ a successor beautiful cardinal, if it is the first beautiful cardinal $> \chi$, for some χ; limit otherwise (see 2.4(6)).

By Silver [25]

2.4. THEOREM. (1) *If κ is a beautiful cardinal, then also in the universe L ($=$ the class of constructible sets, introduced by Godel) κ is beautiful.*

(2) *If κ is the first cardinal such that $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ then κ is beautiful (hence is the first beautiful cardinal $> \chi$), and is strongly inaccessible, but is not weakly compact.*

(3) *The class of beautiful cardinals is closed, and every member is strong limit, and moreover is limit of weakly compact cardinals, provided it is uncountable.*

(4) *$\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ iff every model M with universe κ and χ relations and functions (finitary, of course) has a submodel N and a non-trivial monomorphism $f: N \to N$ (i.e., f is not the identity). Notice that by using Skolem functions, we can assume that N is an elementary submodel of M, and f is elementary embedding.*

(5) *Suppose κ is beautiful $> \aleph_0$, M a model with universe κ and $< \kappa$ relations and functions. Then for any $\alpha < \kappa$ there are $\alpha(n) < \kappa$ (for $n < \omega$) such that*

(a) $\alpha < \alpha(0) < \alpha(1) < \cdots < \alpha(n) < \cdots$,

(b) *for every n, $\langle \alpha(0), \cdots, \alpha(n)\rangle$, $\langle \alpha(1), \cdots, \alpha(n+1)\rangle$ realizes the same type in M over $\alpha(0)$, i.e., for every formula $\varphi(x_0, \cdots, x_n, y_0, \cdots, y_{k-1})$ in the language of M, and $\gamma_0, \cdots, \gamma_{k-1} < \alpha(0)$,*

$$M \models \varphi[\alpha(0), \cdots, \alpha(n), \gamma_0, \cdots, \gamma_{k-1}] \qquad \textit{iff}$$

$$M \models \varphi[\alpha(1), \cdots, \alpha(n+1), \gamma_0, \cdots, \gamma_{k-1}].$$

(6) *In (5), if κ is a successor beautiful, $C \subseteq \kappa$ is closed unbounded, then we can choose the $\alpha(n)$'s in C. A κ with this property is called strongly beautiful; singular*

cardinals are not strongly beautiful, but if κ is regular and $\{\lambda < \kappa : \lambda$ strongly beautiful$\}$ is stationary, then κ is strongly beautiful.

2.5. THEOREM. *Suppose* $\mathbf{Q}$ *is not* χ*-narrow* ($\chi \geqq \aleph_0$) *and* $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ *fails. Then for some* α, $\mathcal{P}_\alpha(\mathbf{Q})$ *is not* κ*-narrow.*

REMARK. Why have we not done it this time for well-ordering as well? Because it fails, see 3.1.

PROOF. Let $F : D\ \mathrm{Seq}_{<\omega}(\kappa) \to \chi$ exemplify the failure of $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$. By easy changes (which retain its being a counterexample) we can assume:

(a) From $F(\eta)$ we can compute $l(\eta)$ and $F(\langle \eta(k), \cdots, \eta(m-1)\rangle)$, $F(\langle \eta(k), \eta(m)\rangle)$ for $k \leqq m \leqq l(\eta)$.

(b) From $F(\langle i, j\rangle)$ we can compute the truth value of $i < j$, $j < i$.

We define by induction on n a set $C^n \subseteq D\ \mathrm{Seq}_n(\kappa)$:

$$C^0 = D\ \mathrm{Seq}_0(\kappa), \qquad C^1 = D\ \mathrm{Seq}_1(\kappa),$$
$$C^{n+1} = \{\eta \in D\ \mathrm{Seq}_{n+1}(\kappa) : \eta \restriction n \in C^n,\ \eta^- \in C^n \text{ and } F(\eta \restriction n) = F(\eta^-)\}.$$

Let $C = \bigcup_n C^n$.

It is easy to prove that each $\eta \in C$ is monotonic (increasing or decreasing) (by induction on n, using assumption (b)). It is also clear that C is closed under initial segments, and also $\eta \in C \Rightarrow \eta^- \in C$.

A little less trivial fact on C is that it contains no set of the form $\{\eta \restriction l : l < \omega\}$ where $\eta \in D\ \mathrm{Seq}_\omega(\kappa)$. If η is a counterexample, it is monotonic (by what was said above), but as it is infinite, it is necessarily increasing. Checking the definition of C^n, we see that $\eta(0) < \eta(1) < \cdots$ contradicts the choice of F (as exemplifying that $\kappa \overset{w}{\rightarrow} (\omega)_\chi^{<\omega}$ fails). Hence there is no such η.

Let $B = \{\eta \in D\ \mathrm{Seq}_{<\omega}(\kappa) : \eta \restriction (l(\eta)-1) \in C \text{ but } \eta \notin C\}$. Let $\{t_i : i < \chi\}$ be χ pairwise incomparable elements of $\mathbf{Q}$, and define for $\eta \in B$, $q_\eta = t_{F(\eta \restriction (l(\eta)-1))}$. We want to apply Lemma 1.12(1). By the conclusion of 1.12(1), for some α, $\mathcal{P}_\alpha(\mathbf{Q})$ is not κ-D-well-ordered, i.e. not κ-narrow, just what we need. So we have to check the assumptions of 1.12(1). We will show that B is a κ-D-barrier and for no $n, \nu \in B$, $\eta \mathbf{R}_D^1 \nu$, $q_\eta \leqq q_\nu$.

FACT. Assume $\eta, \nu \in B$, $\eta \mathbf{R}_D^1 \nu$. We have to show that not $q_\eta \leqq q_\nu$.

Suppose $q_\eta \leqq q_\nu$, so $t_{F(\eta \restriction (l(\eta)-1))} \leqq t_{F(\nu \restriction (l(\nu)-1))}$, so by the choice of the t_i's (as pairwise incomparable) necessarily $F(\eta \restriction (l(\eta)-1)) = F(\nu \restriction (l(\nu)-1))$. So by assumption (a) in the beginning of the proof, $l(\eta) = l(\nu)$, and let $l(\eta) = n+1$. So $\eta \restriction n$, $\nu \restriction n \in C^n$, $F(\eta \restriction n) = F(\nu \restriction \eta)$ so by the definition of C^{n+1}, $\eta = (\eta \restriction n)^\wedge \langle \nu(n-1)\rangle \in C^{n+1}$. But $\eta \in C^{n+1} \subseteq C$ contradicts $\eta \in B$, so we finish.

FACT. B is a κ-D-barrier.

Notice that $\eta \in B$, implies $l(\eta) \geqq 2$.

By the definition of B, clearly for no $\eta, \nu \in B$, $\eta \lessdot \nu$. Also for every $\eta \in D\ \mathrm{Seq}_\omega(\kappa)$ we proved above that for some $k < \omega$, $\eta \restriction k \not\in C$; if k is minimal, clearly $\eta \restriction k \in B$. Now we have to prove that $\eta \in B$, $\nu \lessdot \eta^-$ implies $\nu \not\in B$, but if $\eta = \langle \alpha_0, \cdots, \alpha_{k-1} \rangle$, then $\langle \alpha_0, \cdots, \alpha_{k-2} \rangle \in C$ (by the definition of B) hence $\langle \alpha_1, \cdots, \alpha_{k-2} \rangle \in C$. (As mentioned above $\nu \in C \Rightarrow \nu^- \in C$.) So $\nu \lessdot \eta^-$ means $\nu \leqq \langle \alpha_1, \cdots, \alpha_{k-2} \rangle$ implies $\nu \in C$ implies $\nu \not\in B$.

Trivially $\mathrm{Dom}\, B = \kappa$ and B has at least two members, so by definition B is a κ-D-barrier.

2.6. CONCLUSION. For any quasi-order $\mathbf{Q}$, the first infinite cardinal κ for which $\mathbf{Q}$ is κ-D-bqo is beautiful.

REMARK. We could have proved directly κ is strongly inaccessible.

PROOF. If $\kappa \not\xrightarrow{w} (\omega)^{<\omega}_\chi$, $\chi < \kappa$, and $\kappa > \aleph_0$, $\chi \geqq \aleph_0$, *then* by κ's definition $\mathbf{Q}$ is not χ-D-bqo, hence by 1.11 for some α, $\mathcal{P}_\alpha(\mathbf{Q})$ is not χ-D-well-ordered ($\equiv$ not χ-narrow), hence by 2.5 for some β, $\mathcal{P}_\beta(\mathcal{P}_\alpha(\mathbf{Q}))$ is not κ-narrow ($\equiv$ not κ-D-well-ordered). But by 1.8, as $\mathbf{Q}$ is κ-D-bqo, $\mathcal{P}_\beta(\mathcal{P}_\alpha(\mathbf{Q}))$ is κ-D-bqo, contradiction.

2.7. DEFINITION. Let $X \in \{I, D\}$, $\lambda \geqq 1$.

(1) The quasi-order $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo *if* for every κ-X-barrier of depth $\leqq \alpha$, B and function $F : B \to \lambda$ and $q_\eta \in \mathbf{Q}$ for $\eta \in B$, *there are* $\eta \in B$, $\nu \in B$ such that $F(\eta) = F(\nu)$, $\eta \mathbf{R}^1_X \nu$ and $q_\eta \leqq q_\nu$.

(2) $\mathbf{Q}$ is $[\kappa; \lambda]$-X-bqo iff $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo for every $\alpha < \lambda^+$. $\mathbf{Q}$ is $[\kappa, \alpha]$-X-bqo iff $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo for every $\lambda < \kappa$, and $\mathbf{Q}$ is $[\kappa]$-X-bqo iff $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo for every $\alpha < \kappa^+$, $\lambda < \kappa$.

(3) In all the above definitions we omit the letter X (or I or D) if the two versions with I and with D are equivalent.

2.8. CLAIM. (1) In all versions of bqo, the I version implies the D version.

(2) Suppose $\kappa \leqq \kappa'$, $\alpha \geqq \alpha'$, $\lambda \geqq \lambda'$, then:

(a) $\mathbf{Q}$ is (κ, α)-X-bqo implies $\mathbf{Q}$ is (κ', α')-X-bqo.

(b) $\mathbf{Q}$ is κ-X-bqo implies $\mathbf{Q}$ is κ'-X-bqo.

(c) $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo implies $\mathbf{Q}$ is $[\kappa', \alpha'; \lambda']$-$X$-bqo.

(d) $\mathbf{Q}$ is $[\kappa; \lambda]$-X-bqo implies $\mathbf{Q}$ is $[\kappa'; \lambda']$-X-bqo.

(3) If $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo ($\lambda > 0$, of course) then $\mathbf{Q}$ is (κ, α)-X-bqo; and if $\mathbf{Q}$ is $[\kappa; \lambda]$-X-bqo then $\mathbf{Q}$ is κ-X-bqo.

(4) If $\lambda \geqq \kappa$ then no $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo.

PROOF. Trivial.

2.9. CLAIM. (1) $\mathbf{Q}$ is $[\kappa,\alpha;1]$-X-bqo iff $\mathbf{Q}$ is (κ,α)-X-bqo; and if $\lambda<\operatorname{cf}\kappa$ then $\mathbf{Q}$ is $[\kappa,1;\lambda]$-X-bqo iff $\mathbf{Q}$ is $(\kappa,1)$-X-bqo; and if $\operatorname{cf}\kappa\leqq\lambda<\kappa$ then $\mathbf{Q}$ is $[\kappa,1;\lambda]$-X-bqo iff $\mathbf{Q}$ is $(\kappa',1)$-X-bqo for some $\kappa'<\kappa$.

(2) $\mathbf{Q}$ is $[\aleph_0,\alpha]$-I-bqo iff $\mathbf{Q}$ is $(\aleph_0,\alpha)$-I-bqo.

(3) If κ is weakly compact, then $\mathbf{Q}$ is (κ,n)-I-bqo iff $\mathbf{Q}$ is $I\operatorname{Seq}_n(\kappa)$-$I$-bqo. Also $\mathbf{Q}$ is $[\kappa,n;\lambda]$-I-bqo ($\lambda<\kappa$) iff $\mathbf{Q}$ is $[\kappa,n]$-I-bqo; and if $\mathbf{Q}_i$ ($i<\lambda$) is $[\kappa,n]$-I-bqo ($\lambda<\kappa$) then $\Pi_i\,\mathbf{Q}_i$ is $[\kappa,n]$-I-bqo.

(4) If κ is Ramsey *then*: $\mathbf{Q}$ is $[\kappa]$-I-bqo iff $\mathbf{Q}$ is (κ,n)-I-bqo for every n, also if $\lambda<\kappa$, $\mathbf{Q}_i$ is κ-I-bqo then $\Pi_{i<\lambda}\,\mathbf{Q}_i$ is κ-I-bqo.

(5) In Definition 2.7(1) for infinite λ, we can assume w.l.o.g. that from $F(\eta)$ we can compute $l(\eta)$, the truth value of $\eta(n)<\eta(k)$ for $n,k<l(\eta)$ and the value $F(\langle\eta(l_0),\cdots,\eta(l_k)\rangle)$ for any $k<l(\eta)$, $l_0<l(\eta),\cdots,l_k<l(\eta)$, such that $\langle\eta(l_0),\cdots,\eta(l_k)\rangle\in B$. Instead of $\operatorname{Range}(F)\subseteq\lambda$ we can demand that $|\operatorname{Range}(F)|\leqq\lambda$.

PROOF. (1) By the definitions.

(2) By Nash-Williams [19].

(3), (4) Easy (see 2.2).

(5) Trivial.

2.10. THEOREM. *If κ is beautiful, $\|\mathbf{Q}\|<\kappa$ then $\mathbf{Q}$ is $[\kappa]$-I-bqo. (Hence, by 2.8(1), $\mathbf{Q}$ is also $[\kappa]$-D-bqo.)*

PROOF. For the case $\kappa=\aleph_0$ see [19]. So we assume $\kappa>\aleph_0$.

Let B be a κ-I-barrier, $q_\eta\in\mathbf{Q}$ for $\eta\in B$ and $F:B\to\chi$, $\chi<\kappa$. We define a model M: its universe is κ, and its relations:

$$R^n_{q,j}=\{\eta\in B: l(\eta)=n,\ q_\eta=q,\ F(\eta)=j\}\quad\text{for } q\in Q,\quad j<\chi,\quad n<\omega.$$

So by 2.4(5) there is $\eta\in I\operatorname{Seq}_\omega(\kappa)$ such that for each n, the sequences $\eta\restriction n$ and $(\eta^-)\restriction n$ realize in M the same type. As B is an I-barrier for some n, $\eta\restriction n\in B$. For some q,j $M\models R^n_{q,j}[\eta\restriction n]$, hence $M\models R^n_{q,j}[(\eta^-)\restriction n]$: thus $(\eta^-)\restriction n\in B$, $F((\eta^-)\restriction n)=F(\eta\restriction n)$ and $q_{(\eta^-)\restriction n}=q_{\eta\restriction n}$. So $\eta\restriction n$, $(\eta^-)\restriction n$ are as required from η, ν in the definition of $[\kappa]$-I-bqo.

2.11. THEOREM. (1) *For any $X\in\{I,D\}$, $\mathbf{Q}$, κ, α and λ, $1<\lambda<\kappa$, the following conditions are equivalent (where $\lambda=\lambda(0)\lambda(1)$):*

(a) $\mathbf{Q}$ *is* $[\kappa,\alpha;\lambda]$-X-bqo.

(b) $\mathbf{Q}\times(\lambda,=)$ *is* (κ,α)-X-bqo (*note that* $(\lambda,=)$ *is a quasi-order*).

(c) $\mathbf{Q}\times(\lambda(0),=)$ *is* $[\kappa,\alpha;\lambda(1)]$-$X$-bqo.

(2) *If* $\mathbf{Q}$ *is* $[\kappa, \alpha+1; (\lambda(0)+1)\lambda(1)]$-$X$-bqo *then* $\mathbf{Q}^{\lambda(0)}$ (*this is a power of* $\mathbf{Q}$) *is* $[\kappa, \alpha; \lambda(1)]$-$X$-bqo.

(3) *For* X, $\mathbf{Q}$, κ, α *and* λ *as above, the following are equivalent, with two possible exceptions,* (i) $(\forall q, q_2 \in Q)(q_1 \leqq q_2 \vee q_2 \leqq q_1)$ *and* $\lambda = \lambda(0) = 2$, (ii) $(\forall q_1, q_2 \in Q)$ $q_1 \leqq q_2$, (iii) $\lambda(0) < 2$ *or* $\lambda(0) = 2$. $\mathbf{Q}$ *has no three incomparable members*:

(a) $\mathbf{Q}$ *is* $[\kappa; \lambda]$-X-bqo.

(b) $\mathbf{Q} \times (\lambda, =)$ *is* κ-X-bqo.

(c) $\mathbf{Q} \times (\lambda(0)+1, =)$ *is* $[\kappa; \lambda(1)]$-X-bqo.

(d) $\mathbf{Q}^{\lambda(0)}$ *is* $[\kappa; \lambda(1)]$-X-bqo.

(4) *For infinite* λ, $\mathbf{Q}$ *is* $[\kappa; \lambda]$-X-bqo *iff* $\mathbf{Q}^{\lambda}$ *is* $[\kappa; \lambda]$-X-bqo.

REMARK. For $\lambda = 1$, the theorem holds trivially by 2.9(1).

PROOF. (1) Clearly (b) is a particular case of (c) (for $\lambda(1) = 1$, see 2.9(1)). So it suffices to prove (a) $\Rightarrow$ $(c)_{\lambda(0),\lambda(1)}$ $\Rightarrow$ (a) (for any $\lambda(0)$, $\lambda(1)$).

(a) $\Rightarrow$ $(c)_{\lambda(0),\lambda(1)}$

So we assume $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo and B is a κ-X-barrier of depth $\leqq \alpha$, $t_\eta = (q_\eta, j_\eta) \in \mathbf{Q} \times (\lambda(0), =)$ and $F: B \to \lambda(1)$. We have to prove that for some $\eta, \nu \in B$, $\eta \mathbf{R}^1_X \nu$, $F(\eta) = F(\nu)$ and $t_\eta \leqq t_\nu$. Let $p: \lambda(0) \times \lambda(1) \to \lambda$ be one to one and onto λ (it exists as $\lambda(0)\lambda(1) = \lambda$), and $p_1: \lambda \to \lambda(0)$, $p_2: \lambda \to \lambda(1)$ be such that for every $i < \lambda$, $i = p(p_1(i), p_2(i))$. We define a function $F': B \to \lambda$ by $F'(\eta) = p(j_\eta, F(\eta))$ hence:

$$(*) \qquad F'(\eta) = F'(\nu) \quad \text{iff } F(\eta) = F(\nu), \quad j_\eta = j_\nu.$$

As $\mathbf{Q}$ is $[\kappa, \alpha; \lambda]$-$X$-bqo, B a κ-X-barrier of depth $\leqq \alpha$, $q_\eta \in Q$ for $\eta \in B$, and $F': B \to \lambda$, *clearly* for some $\eta \in B$, $\nu \in B$, $\eta \mathbf{R}^1_X \nu$, $F'(\eta) = F'(\nu)$ and $q_\eta \leqq q_\nu$. Now as $F'(\eta) = F'(\nu)$ necessarily $F(\eta) = F(\nu)$ and also $j_\eta = j_\nu$, hence $t_\eta = (q_\eta, j_\eta) \leqq (q_\nu, j_\nu) = t_\nu$. So we find $\eta \in B$, $\nu \in B$ such that $\eta \mathbf{R}^1_X \nu$, $t_\eta \leqq t_\nu$, $F(\eta) = F(\nu)$, as required.

$(c)_{\lambda(0),\lambda(1)}$ $\Rightarrow$ (a)

So we assume $\mathbf{Q} \times (\lambda(0), =)$ is $[\kappa, \alpha; \lambda(1)]$-$X$-bqo, B is a κ-X-barrier of depth $\leqq \alpha$ and $q_\eta \in \mathbf{Q}$ for $\eta \in B$, $F: B \to \lambda$. We have to prove that for some $\eta, \nu \in B$, $\eta \mathbf{R}^1_X \nu$, $F(\eta) = F(\nu)$ and $q_\eta \leqq q_\nu$.

We use p, p_1, p_2 defined above. Let $t_\eta = (q_\eta, p_1(F(\eta))) \in \mathbf{Q} \times (\lambda(0), =)$ and $F': B \to \lambda(1)$ be $F'(\eta) = p_2(F(\eta))$. So as $\mathbf{Q} \times (\lambda(0), =)$ is $[\kappa, \alpha; \lambda(1)]$-$X$-bqo there are $\eta, \nu \in B$ such that $\eta \mathbf{R}^1_X \nu$, $t_\eta \leqq t_\nu$ and $F'(\eta) = F'(\nu)$. Hence $q_\eta \leqq q_\nu$, $p_1(F(\eta)) = p_1(F(\nu))$ and $p_2(F(\eta)) = p_2(F(\nu))$. So $F(\eta) = F(\nu)$ and we finish.

(2) Let B be a κ-X-barrier of depth $\leqq \alpha$, $F: B \to \lambda(1)$ and $\bar{q}_\eta =$

$\langle q^i_\eta : i < \lambda(0)\rangle \in \mathbf{Q}^{\lambda(0)}$ (so $q^i_\eta \in Q$) for $\eta \in B$. We suppose for no $\eta, \nu \in B$ does $\eta \mathbf{R}^1_X \nu$, $\bar{q}_\eta \leqq \bar{q}_\nu$, $F(\eta) = F(\nu)$. Let B' be as in 1.9, and we shall define $t_\sigma \in Q$, $F'(\sigma) < \lambda = (\lambda(0)+1)\lambda(1)$ for every $\sigma \in B'$. So let $\sigma \in B'$. Then there are unique $\eta, \nu \in B$, $\sigma = \eta \cup^* \nu$, $\eta \mathbf{R}^1_X \nu$. If $F(\eta) \neq F(\nu)$ we let $i(\sigma) = \lambda(0)$, $t_\sigma = q^{i(\sigma)}_\eta$, $F'(\sigma) = p(i(\sigma), F(\eta))$. But if $F(\eta) = F(\nu)$ then by an assumption above not $\bar{q}_\eta \leqq \bar{q}_\nu$, hence by the definition of the order of $\mathbf{Q}^{\lambda(0)}$, for some $i(\sigma) < \lambda(0)$, not $q^i_\eta \leqq q^i_\nu$. In this case we let $t_\sigma = q^{i(\sigma)}_\eta$, $F'(\sigma) = p(i(\sigma), F(\eta))$.

So we have a κ-X-barrier B' of depth $\leqq \alpha + 1$ (by 1.9) and $t_\sigma \in Q$ for $\sigma \in B'$ and $F' : B' \to \lambda$. So as $\mathbf{Q}$ is $[\kappa, \alpha + 1; \lambda]$-$X$-bqo there are $\sigma_1, \sigma_2 \in B'$ such that $\sigma_1 \mathbf{R}^1_X \sigma_2$, $t_{\sigma_1} \leqq t_{\sigma_2}$ and $F'(\sigma_1) = F'(\sigma_2)$.

Let $\sigma_l = \eta_l \cup^* \nu_l$, $\eta_l \in B$, $\nu_l \in B$, $\eta_l \mathbf{R}^1_X \nu_l$ for $l = 1, 2$. As $F'(\sigma_1) = F'(\sigma_2)$ and as we know that $p_2(F'(\sigma_l)) = F(\eta_l)$ clearly $F(\eta_1) = F(\eta_2)$. It is also clear that necessarily $\eta_2 = \nu_1$, so $F(\eta_1) = F(\nu_1)$. So $i = i(\sigma_1)$ is $< \lambda(0)$ and not $q^i_{\eta_1} \leqq q^i_{\nu_1}$. On the other hand $F'(\sigma_1) = F'(\sigma_2)$ implies $i = i(\sigma_1) = p_1(F'(\sigma_1)) = p_1(F'(\sigma_2)) = i(\sigma_2)$ hence $q^i_{\eta_1} = t_{\sigma_1} \leqq t_{\sigma_2} = q^i_{\eta_2}$, contradiction.

(3) By the first part (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), and by the second part (a) $\Rightarrow$ (d). Hence it suffices to prove not (b) implies not (d), and for simplicity assume $\lambda(0) + 1 = \lambda$. Let B be a κ-X-barrier, $(q_\eta, \alpha_\eta) \in \mathbf{Q} \times (\lambda, =)$, $\eta \in B$, exemplify not (b), and it suffices to prove $\mathbf{Q}^\lambda$ is not B-X-bqo.

Case (a). In $\mathbf{Q}$ there are two incomparable elements q^0, q^1, $\lambda \geqq 4$. Then let $\nu_\alpha \in {}^{\lambda(0)-\{0\}}2$ $(\alpha < \lambda)$ be distinct (they exist as $2^{\lambda-2} \geqq \lambda$ for $\lambda > 4$) and define $2^{\lambda-2} \geqq \lambda$ for $\lambda \geqq 4$, $\bar{q}_\eta = \langle q^i_\eta : i < \lambda(0)\rangle \in \mathbf{Q}^{\lambda(0)}$, $q^0_\eta = q_\eta$, $q^i_\eta = q^l$ where $l = \nu_{\alpha_\eta}(i)$ for $0 < i < \lambda(0)$. It is easy to check that $\langle \bar{q}_\eta : \eta \in B\rangle$ exemplify not (d).

Case (b). In $\mathbf{Q}$ there are $q^0 \not\leqq q^1$, $2^{(\lambda-2)/2} \geqq \lambda$ (i.e. $\lambda \geqq 10$). Find $\nu_\alpha \in {}^{\lambda(0)-\{0\}}2$ $(\alpha < \lambda)$ such that $\alpha \neq \beta \to (\exists i)(\nu_\alpha(i) = 0 \wedge \nu_\beta(i) = 1)$, and proceed as above.

Case (c). In $\mathbf{Q}$ there is a strictly decreasing sequence of length λ, $10 > \lambda > 3$ $(q^i \in \mathbf{Q}\ (i < \lambda),\ q^i \not\leqq q^j$ for $i < j)$. Let $\nu_\alpha(1) = \alpha$, $\nu_\alpha(2) = \lambda - 1 - \alpha$, and proceed as before.

Case (d). In $\mathbf{Q}$ there are three incomparable members, q^0, q^1, q^2, and $\lambda = 3$. We define $\nu_\alpha(1) = q^\alpha$ for $\alpha < \lambda$ and proceed as before.

Let us show that either one of the cases apply or one of the exceptions apply. We can assume that exception (iii) does not apply, hence by Case (d) we can assume $\lambda \geqq 4$.

If (a) is not the case, we can assume that $\mathbf{Q}$ is a linear order (by considering $\mathbf{Q}$ modulo the relation $q_1 \leqq q_2 \wedge q_2 \leqq q_1$). If (b) is not the case and we are not in exception (ii), we can assume further that $\lambda < 10$, and since we are not in exception (i), $\lambda > 3$. Thus, if (c) is not the case, we can assume that $\mathbf{Q}$ is a finite ordinal and $\lambda < 10$, but then (b) holds by 2.10.

(4) Easy, by part (3).

2.11 A. REMARK. From the proof it is clear that, e.g., for $3 \leqq \lambda < \omega$, $\mathbf{Q}^\lambda$ is κ-X-bqo iff $\mathbf{Q}^3$ is κ-X-bqo.

2.12. THEOREM. (1) *Suppose* $a_i, b_i \in \mathbf{Q}$ *for every* $i < \kappa$ *and*

(i) *for* $i < j < \kappa$, $a_i \not\leqq a_j$, $b_i \not\geqq b_j$;

(ii) *for* $i, j < \kappa$, a_i *and* b_j *are incomparable.*

Then $\mathcal{P}(\mathbf{Q})$ *is not* κ*-narrow.*

(2) *If for some* α, $\mathcal{P}_\alpha(\mathbf{Q})$ *contains* a_i, b_i *as above then* $\mathcal{P}_{\alpha+1}(\mathbf{Q})$ *is not* κ*-narrow hence* $\mathbf{Q}$ *is not* κ-D-bqo.

PROOF. (1) Let $t_i = \{a_i, b_i\} \in \mathcal{P}(Q)$ for $i < \kappa$. It is easy to check that for $i \neq j < \kappa$, not $t_i \leqq t_j$.

(2) Easy.

2.13. THEOREM. (1) *If* $\mathbf{Q}$ *is* $[\kappa, \alpha + 1; \lambda + 4]$-$D$-bqo *then* $\mathbf{Q}$ *is* $[\kappa, \alpha; \lambda]$-$I$-bqo, *provided that* $\alpha \geqq \kappa$.

(2) *If* λ *is infinite,* $\mathbf{Q}$ *is* $[\kappa; \lambda]$-D-bqo *iff* $\mathbf{Q}$ *is* $[\kappa; \lambda]$-I-bqo (*so we can omit* I *and* D).

PROOF. (1) Let B' be a κ-I-barrier of depth $\leqq \alpha$, $F' : B' \to \lambda$ and $q'_\eta \in Q$, $\eta \in B'$; we want to show that for some $\eta, \nu \in B'$, $\eta \mathbf{R}^1_I \nu$, $F'(\eta) = F'(\nu)$ and $q'_\eta \leqq q'_\nu$. Define $B = \{\eta^\wedge\langle i\rangle : \eta \in B',\ \eta^\wedge\langle i\rangle \in I\,\mathrm{Seq}_{<\omega}(\kappa)\}$, and for $\eta^\wedge\langle i\rangle \in B$ define $F(\eta^\wedge\langle i\rangle) = F'(\eta)$ and $q_{\eta^\wedge\langle i\rangle} = q'_\eta$. Then B is a κ-I-barrier of depth $\leqq \alpha + 1$, and it suffices to show that for some $\eta, \nu \in B$, $\eta \mathbf{R}^1_I \nu$, $F(\eta) = F(\nu)$ and $q_\eta \leqq q_\nu$. For B we can apply 0.8 to obtain the κ-D-barrier $B^* = B \cup B^D$. Define $F^* : B^* \to \lambda + 4$ as follows (using the notations of 0.9):

$$F^*(\eta) = \begin{cases} F(\eta) & \text{if } \eta \in B, \\ \lambda & \text{if } \eta \in B^D_{n,u} \text{ for some even } n > 3, \\ \lambda + 1 & \text{if } \eta \in B^D_{n,u} \text{ for some odd } n \geqq 3, \\ \lambda + 2 & \text{if } \eta \in B^D_{n,d} \text{ for some even } n > 3, \\ \lambda + 3 & \text{if } \eta \in B^D_{n,d} \text{ for some odd } n \geqq 3. \end{cases}$$

Define $q^*_\eta = q_\eta$, for $\eta \in B$, and $q^*_\eta =$ some arbitrary element of $\mathbf{Q}$, for $\eta \in B^D$. Since by 0.10, $\mathrm{Dp}(B^*) \leqq \alpha + 1$, we can find $\eta, \nu \in B^*$, $\eta \mathbf{R}^1_D \nu$, $F^*(\eta) = F^*(\nu)$ and $q^*_\eta \leqq q^*_\nu$. Since $\eta \mathbf{R}^1_D \nu$ and $F^*(\eta) = F^*(\nu)$, by 0.9 we must have $\eta, \nu \in B$. By our definitions $\eta \mathbf{R}^1_I \nu$, $F(\eta) = F(\nu)$ and $q_\eta \leqq q_\nu$, so we have finished.

(2) One direction follows by part (1), the other follows by 2.8 (1).

2.14. CONCLUSION. Suppose $\lambda \geqq \aleph_0$, $\mathbf{Q}$ a quasi-order. The first cardinal κ for which $\mathbf{Q}$ is $[\kappa; \lambda]$-bqo is beautiful.

2.15. THEOREM. *Theorem* 1.8, *which we proved for* κ-X-bqo, *holds for* $[\kappa;\lambda]$-X-bqo $(0<\lambda<\kappa)$ *and* $[\kappa]$-X-bqo *as well, with* $\mathcal{P}_\alpha$ *replaced by* $\mathcal{P}^0_\alpha$.

PROOF. We are given in addition to $A_\eta \in \mathcal{P}^0_\alpha(\mathbf{Q})(\eta \in B)$ also $F : B \to \lambda$. We define by induction on n, B_n, $F_n : B_n \to \lambda$ and t^n_η (for $\eta \in B_n$).

For $n=0$: $B_0 = B$, $F_0 = F$, $t^0_\eta = A_\eta$ (for $\eta \in B_0$).

For $n+1$: We let

$$B_{n+1} = \{\eta \cup^* \nu : \eta \in B_n,\ \nu \in B_n,\ \eta \mathbf{R}^1_X \nu,\ t^n_\eta \notin \mathbf{Q}\} \cup \{\eta : \eta \in B_n \text{ and } t^n_\eta \in \mathbf{Q}\}.$$

$$F_{n+1}(\eta \cup^* \nu) = F_n(\eta) \qquad \text{for } \eta \cup^* \nu \in B_{n+1},\quad \eta \mathbf{R}^1_X \nu,$$

$$F_{n+1}(\eta) = F_n(\eta) \qquad \text{for } \eta \in B_n,\quad t^n_\eta \in \mathbf{Q},$$

$$t^{n+1}_\sigma = \begin{cases} t^n_\eta & \text{if } \sigma = \eta \in B_n,\quad t^n_\eta \in \mathbf{Q}; \\ \text{any member of} & \text{if } \sigma = \eta \cup^* \nu \in B_{n+1}, \\ \quad Tc(t^n_\eta) \in \mathbf{Q} & \quad \eta \mathbf{R}^1_X \nu,\quad F_n(\eta) \neq F_n(\nu); \\ \text{any } t \in t^n_\eta \text{, such that} & \text{if } \sigma = \eta \cup^* \nu \in B_{n+1}, \eta \mathbf{R}^1_x \nu, \\ \quad (\forall s \in t^n_\nu)\ (\text{not } t \leqq s) & \quad F_n(\eta) = F_n(\nu). \\ \quad \text{and if } t^n_\nu \in \mathbf{Q},\ t \not\leqq t^n_\nu & \end{cases}$$

At last

$$B^* = \{\eta : \text{for some } n, \eta \in B_n, t^n_\eta \in \mathbf{Q}\}.$$

In the rest of the proof, we have to introduce only minor changes.

§3. Examples

3.1. THEOREM. *Suppose* $\mathbf{Q}$ *is a linear order.*

(1) *If in* $\mathbf{Q}$ *there is no descending sequence of length* κ, *then* $\mathbf{Q}$ *is* κ-I-bqo.

(2) *If* α *is an ordinal, then* $\mathcal{P}_\alpha(\mathbf{Q})$ *has no two incomparable elements.*

(3) $\mathbf{Q}$ *has no descending sequence of length* κ *iff* $\mathbf{Q}$ *is* κ*-well-ordered.*

(4) $\mathbf{Q}$ *is* $\aleph_0$-D-bqo.

REMARK. We can replace here κ by any limit ordinal.

PROOF. (1) Suppose B is a κ-I-barrier, $q_\eta \in Q$ for $\eta \in B$, and $\eta, \nu \in B$, $\eta \mathbf{R}^1_I \nu$ implies not $q_\eta \leqq q_\nu$. We shall find a descending sequence of members of $\mathbf{Q}$ of length κ, thus finishing. W.l.o.g. $\operatorname{Dom} B = \kappa$.

We now define by induction on $\alpha < \kappa$ a sequence $\eta_\alpha \in B$ such that

(i) for $\beta < \alpha$, $k < l(\eta_\beta)$, $\eta_\beta(k) < \eta_\alpha(0)$,

(ii) $\eta_\alpha(l)$ is $< \alpha + \omega$.

For any α, let $\gamma_\alpha = \bigcup\{\eta_\beta(l)+1 : \beta < \alpha,\ l < l(\eta_\beta)\}$. It is easy to check that $\gamma_\alpha < \alpha + \omega$. Now $\langle \gamma_\alpha, \gamma_\alpha + 1, \gamma_\alpha + 2, \cdots \rangle \in I\,\mathrm{Seq}_\omega(\kappa)$, hence some initial segment $\eta_\alpha = \langle \gamma_\alpha + l : l < l(\eta_\alpha)\rangle \in B$. Clearly η_α is as required.

We shall now prove that for $\beta < \alpha$, $q_{\eta_\alpha} > q_{\eta_\beta}$, thus trivially finishing. Now we apply (for specific $\beta < \alpha$) Claim 0.5. By it, there are $k < \omega$, $\sigma_0, \cdots, \sigma_k$ such that $\sigma_l \in B$, $\sigma_0 = \eta_\beta$, $\sigma_k = \eta_\alpha$, $\sigma_l \mathbf{R}^1_I \sigma_{l+1}$. For each l, by the choice of the q_η's, clearly not $q_{\sigma_l} \leqq q_{\sigma_{l+1}}$. But $\mathbf{Q}$ is a linear order, hence $q_{\sigma_l} > q_{\sigma_{l+1}}$. Thus

$$q_{\eta_\beta} = q_{\sigma_0} > q_{\sigma_1} > \cdots > q_{\sigma_k} = q_{\eta_\alpha},$$

so $q_{\eta_\beta} > q_{\eta_\alpha}$, as required.

(2) By induction on α we can prove that for every $q^0 \in \mathcal{P}_\alpha(\mathbf{Q})$ there is $q^1 \in \mathcal{P}_1(\mathbf{Q})$ such that $(\forall q, q')\ q \in q^1 \wedge q' \leqq q \wedge q \in \mathbf{Q} \wedge q' \in \mathbf{Q} \Rightarrow q' \in q^1$.

(3) By definition.

(4) By (2) and 1.11.

3.2. Conclusion. (1) For every κ, there is a quasi-order $\mathbf{Q}_\kappa$ which is κ'-I-bqo iff $\kappa' \geqq \kappa$.

(2) For any ordinal $\alpha > 0$ there is a linear order $\mathbf{Q}_\alpha$, satisfying: there is a descending sequence of length β from $\mathbf{Q}_\alpha$ iff $\beta < \alpha$.

Remark. Compare this with 2.5, 2.6.

Proof. (1) By 3.1 and part (2).

(2) By induction on α. If there are $\beta, \gamma < \alpha$ such that $\alpha = \beta + \gamma$, let $\mathbf{Q}_\alpha = \mathbf{Q}_\gamma + \beta^*$ (β^* denotes the inverse order). Otherwise, let $\mathbf{Q}_\alpha = \Sigma_{\beta<\alpha}\beta^*$.

3.3. Claim. If $\mathbf{Q} = \bigcup_{j<\lambda} \mathbf{Q}_j$, $\lambda < \kappa$, $\kappa > \aleph_0$ beautiful, each $\mathbf{Q}_j$ is linear $\aleph_0$-well-ordered, then $\mathbf{Q}$ is $[\kappa]$-I-bqo.

Proof. Similar to 2.10.

Remark. $\mathbf{Q} = \bigcup \mathbf{Q}_j$ implies $\mathbf{Q}_j = \mathbf{Q} \restriction |\mathbf{Q}_j|$ but for $q_{j_1} \in \mathbf{Q}_{j_1}$, $j_1 \neq j_2$, $q_{j_2} \in \mathbf{Q}_{j_2}$, we do not restrict the order between q_{j_1}, q_{j_2}.

3.4. Claim. Suppose κ is beautiful and singular. Then there is a quasi-order $\mathbf{Q}$ which is for $X = I, D$, κ-X-bqo, but not $[\kappa, 1; \mathrm{cf}\,\kappa]$-$X$-bqo, and not κ'-X-bqo for $\kappa' < \kappa$ (not even κ'-X-well-ordered).

Proof. Let $\kappa = \Sigma_{i<\mu}\kappa_i$, $\mu = \mathrm{cf}\,\kappa < \kappa_i < \kappa$, $Q = \bigcup_{i<\mu} Q_i$, the Q_i's pairwise disjoint, $|Q_i| = \kappa_i$, and for $a, b \in Q$

$$a \leqq b \quad \textit{iff} \quad a = b \quad \text{or} \quad a \in Q_i, \quad b \in Q_j, \quad i < j.$$

Let $\mathbf{Q} = (Q, \leqq)$.

FACT A. $\mathbf{Q}$ is not $[\kappa, 1, \operatorname{cf} \kappa]$-$X$-bqo.

Let $B = \{\langle i\rangle : i < \kappa\}$, $F(\langle i\rangle) = \min\{\alpha : i < \kappa_\alpha\}$, $q_{\langle i\rangle} \in Q_{F(\langle i\rangle)}$, and $i \neq j \Rightarrow q_{\langle i\rangle} \neq q_{\langle j\rangle}$.

FACT B. $\mathbf{Q}$ is not κ'-X-bqo nor κ'-X-well-ordered for $\kappa' < \kappa$.

For each $\kappa' < \kappa$ for some $\alpha < \mu$, $\kappa' < \kappa_\alpha$, and use Q_α.

FACT C. $\mathbf{Q}$ is κ-X-bqo.

Let B be a κ-X-barrier, $\operatorname{Dom} B = \kappa$; $q_\eta \in \mathbf{Q}$ for $\eta \in B$ be a counterexample. Now the relation $R(x, y) =^{\mathrm{df}}$ "not $x < y$" is transitive. Hence, as in the proof of 3.1(1), we can prove that if $\eta(l) < \nu(n)$ for all $l < l(\eta)$, $n < l(\nu)$ and $\eta, \nu \in B$ then "not $q_\eta < q_\nu$". For some m, $\eta_0 = \langle 0, 1, 2, \cdots, m-1\rangle \in B$, and let $q_{\eta_0} \in Q_{\alpha_0}$, so $\nu \in B$, $\nu(0) \geqq m$ implies "not $q_{\eta_0} < q_\nu$". Hence $q_\nu \in \bigcup_{\alpha \leqq \alpha_0} Q_\alpha$. $B' = B \restriction (\operatorname{Dom} B - m)$ is a κ-X-barrier, and $|\bigcup_{\alpha \leqq \alpha_0} Q_\alpha| < \kappa$, so by 2.10 for some $\eta, \nu \in B' \subseteq B$, $\eta\, \mathbf{R}^1_X \nu$, $q_\eta \leqq q_\nu$, contradiction.

3.5. CLAIM. If $\kappa > \aleph_0$ is not weakly compact (e.g., any successor beautiful cardinal is like that by 2.4 (2)) then some linear order $\mathbf{I}$ is κ-I-bqo but not $[\kappa; 2]$-I-bqo (nor $[\kappa; 2]$-D-bqo).

PROOF. As κ is not weakly compact, there is a linear order $\mathbf{I}$ with $|\mathbf{I}| = \kappa$ but no descending nor ascending sequence of length κ. (See e.g. [6].) We can assume that there is $f : \mathbf{I} \to \mathbf{I}$, which is an anti-isomorphism (i.e., $x < y \Leftrightarrow f(x) > f(y)$; we can assume this as $I + I^*$ satisfies this).

In $\mathcal{P}(\mathbf{I} \times \{2, =\})$, there are κ pairwise incomparable elements: $\{t_x : x \in I\}$ where $t_x = \{\langle x, 0\rangle, \langle f(x), 1\rangle\}$.

REMARK. Clearly $\mathbf{I}^2$ is not κ-$\mathcal{D}$-bqo.

§4. More information

4.1. DEFINITION. $\mathcal{P}^1_{<\kappa}(\mathbf{Q}) = (\mathcal{P}_{<\kappa}(\mathbf{Q}), \leqq_1)$ is defined similarly to $\mathcal{P}_{<\kappa}(\mathbf{Q})$ (see 1.3) but the mappings have to be one-to-one.

4.2. THEOREM. *Suppose κ is weakly compact and $\mathbf{Q}$ is κ-well-ordered. Then $\mathcal{P}_{<\kappa}(\mathbf{Q})$ is κ-well-ordered.*

REMARK. Note that if $V = L$, κ is regular not weakly compact, then Jensen proved that there is a κ-Suslin tree T (see [1]). Now it is well-known that even $\mathcal{P}_{<3}(T)$ is not κ-well-ordered, whereas T is.

PROOF. Suppose $S_i \in \mathcal{P}_{<\kappa}(\mathbf{Q})$ $(i < \kappa)$ from a counterexample. For every $i < j$, as not $S_i \leqq S_j$, there is $t_{i,j} \in S_i$, such that for no $s \in S_j$, $t_{i,j} \leqq s$. Let $R = \{(\alpha, \beta, \gamma) : \alpha < \beta < \gamma < \kappa,\ t_{\alpha,\beta} = t_{\alpha,\gamma}\}$, so as κ is weakly compact $\kappa \to (\kappa)^3_2$ so there is $A \subseteq \kappa$, $|A| = \kappa$, such that all increasing triples from A are in R, or all of them are not in R. The second case is impossible, as then for $\alpha \in A$, $\{t_{\alpha,\beta} : \alpha < \beta \in A\}$ is a set of κ distinct members of S_α, but $S_\alpha \in \mathcal{P}_{<\kappa}(\mathbf{Q})$ hence $|S_\alpha| < \kappa$. So for every $\alpha \in A$ there is $t_\alpha \in S_\alpha$ such that $\alpha < \beta \in A$ implies $t_\alpha = t_{\alpha,\beta}$. Let $\alpha < \beta$, $\alpha \in A$, $\beta \in A$, so $t_\alpha = t_{\alpha,\beta} \in S_\alpha$, $t_\beta \in S_\beta$, hence by the choice of $t_{\alpha,\beta}$ not $t_\alpha \leqq t_\beta$. So $\{t_\alpha : \alpha \in A\}$ exemplifies that $\mathbf{Q}$ is not κ-well-ordered.

4.3. DEFINITION. (1) We call a cardinal κ *subtle* if for any sequence $\langle S_\alpha : \alpha < \kappa \rangle$, $S_\alpha \subseteq \alpha$, and closed unbounded subset C of κ, there are $\alpha < \beta$ in C such that $S_\alpha = S_\beta \cap \alpha$.

(2) We call $A \subseteq \kappa$ subtle if we could have chosen α, β in $A \cap C$.

(3) We call κ almost ineffable if every set $A \subseteq \kappa$ which is in the weakly compact filter D^{wc}_κ is subtle, where

$$D^{wc}_\kappa = \{\kappa - S : \text{for some } A \subseteq H(\kappa), \text{ and } \pi^1_1 \text{ sentence } \psi; (H(\kappa), \in, A) \models \psi, \text{ but for no } \alpha \in S, (H(\alpha), \in, A \cap H(\alpha)) \models \psi\}.$$

For the following see Kunen and Jensen [9].

4.4. LEMMA. (1) *A subtle cardinal is strongly inaccessible and even the limit of weakly compact cardinals.*

(2) *Any successor beautiful cardinal is subtle.*

(3) *A cardinal κ is weakly compact iff $\emptyset \notin D^{wc}_\kappa$ iff D^{wc}_κ is a normal filter.*

(4) *Any almost ineffable cardinal is weakly compact and subtle, but bigger than the first cardinal which is weakly compact and subtle.*

(5) *Any ineffable cardinal is the limit of almost ineffable cardinals.*

(6) *A cardinal κ is almost ineffable iff for every two-place function on κ satisfying $f(i,j) < i$ for $0 < i < j$, there are α and $A \subseteq \kappa$, $|A| = \kappa$,* s.t. $[i, j \in A \wedge i < j] \Rightarrow f(i,j) = \alpha$.

4.5. THEOREM. *Suppose κ is almost ineffable and $\mathbf{Q}$ is κ-well-ordered. Then $\mathcal{P}^1_{<\kappa}(\mathbf{Q})$ is κ-well-ordered.*

PROOF. Suppose $S_i \in \mathcal{P}^1_{<\kappa}(\mathbf{Q})$ for $i < \kappa$, so $S_i \subseteq \mathbf{Q}$, $|S_i| < \kappa$. We have to prove that for some $i < j$, $S_i \leqq_1 S_j$, i.e., there is a one-to-one $h : S_i \to S_j$ such that $t \leqq h(t)$ for every $t \in S_i$.

For any $\alpha < \beta$, let

$$S(\alpha, \beta) = \{t \in S_\alpha : \{s \in S_\beta : t \leqq s\} \text{ has cardinality} \leqq |S_\alpha|\}.$$

Notice $S(\alpha,\beta)\subseteq S_\alpha$ and:

$$(*)\qquad \text{if } S(\alpha,\beta)\leqq_1 S_\beta \text{ then } S_\alpha\leqq_1 S_\beta.$$

Because by the assumption there is a one-to-one $h_1: S(\alpha,\beta)\to S_\beta$, $t\leqq h_1(t)$, and we can extend it to $h: S_\alpha\to S_\beta$ by the definition of $S(\alpha,\beta)$.

Now for $\alpha<\beta$, let

$$S^*(\alpha,\beta)=\{s\in S_\beta : \text{for some } t\in S(\alpha,\beta),\ t\leqq s\},$$

$$S^*_\beta=\bigcup_{\alpha<\beta} S^*(\alpha,\beta).$$

Let $S^*_\alpha=\{t^\alpha_i : i<|S^*_\alpha|\}$. Let $\alpha G\beta$ mean that:

(i) $\alpha<\beta<\kappa$,

(ii) $|S^*_\alpha|\leqq|S^*_\beta|$,

(iii) $(\forall i<|S^*_\alpha|)t^\alpha_i\leqq t^\beta_i$,

(iv) $(S_\alpha-S^*_\alpha)\leqq(S_\beta-S^*_\beta)$.

Assume $\alpha G\beta$. Let h be a function exemplifying (iv). If $t\in S_\alpha-S^*_\alpha$ then since $h(t)\notin S^*(\alpha,\beta)$ it follows that $t\notin S(\alpha,\beta)$. This shows that $S(\alpha,\beta)\subseteq S^*_\alpha$. The mapping $t^\alpha_i\mapsto t^\beta_i$ shows that $S^*_\alpha\leqq_1 S^*_\beta$ and by monotonicity properties of this relation it follows that $S(\alpha,\beta)\leqq_1 S_\beta$, hence by $(*)$, $S_\alpha\leqq_1 S_\beta$. Thus it suffices to prove that there are α, β such that $\alpha G\beta$. Let $A=\{\alpha<\kappa : |S^*_\alpha|<\alpha\}$.

Case I. A is stationary in κ

Then by Fodor's theorem $|S^*_\alpha|$ is fixed for κ α's, and by weak compactness of κ, we obtain κ of them such that $\{i : t^\alpha_i\leqq t^\beta_i\}$ is fixed whenever $\alpha<\beta$ are among them. By Theorem 4.2 we can find some $\alpha<\beta$ among them with $(S_\alpha-S^*_\alpha)\leqq(S_\beta-S^*_\beta)$. Since $\mathbf{Q}$ is κ-well-ordered, there is no i with $t^\alpha_i\not\leqq t^\beta_i$. We have shown that $\alpha G\beta$.

Case II. A is not stationary in κ

Then let C be a closed unbounded subset of κ with $A\cap C=\emptyset$. Let $D=\{\lambda<\kappa : \lambda$ is an infinite cardinal and for all $\alpha<\lambda\ |S_\alpha|\leqq\lambda\}$; it is obvious that D is closed, and by Fodor's theorem one can see that it is unbounded. Since by the definitions $|S^*_\beta|\leqq\Sigma_{\alpha<\beta}|S_\alpha|^2$, we have for $\lambda\in D$, $|S^*_\lambda|\leqq\lambda$. Denoting $E=C\cap D$, we obtain that E is closed unbounded and for $\lambda\in E$, $|S^*_\lambda|=\lambda$.

We will finish by showing that for some $\alpha,\beta\in E$, $\alpha G\beta$. Assume that this is false. Define $f(\alpha,\beta)$ for $\alpha,\beta\in E$, $\alpha<\beta$, as follows: if $(S_\alpha-S^*_\alpha)\not\leqq(S_\beta-S^*_\beta)$ then $f(\alpha,\beta)=0$; otherwise since $\alpha G\beta$ is false, there is an $i<|S^*_\alpha|=\alpha$ such that $t^\alpha_i\not\leqq t^\beta_i$, and let $f(\alpha,\beta)$ be the successor of the least such i. Now we use almost ineffability of κ to obtain $F\subseteq E$, $|F|=\kappa$, and j so that for $\alpha<\beta$ in F, $f(\alpha,\beta)=j$

(see 4.4(6); observe that since f is defined on a closed unbounded set, it can always be extended in a manner that yields such F). This, however, is a contradiction, since $j = 0$ would contradict Theorem 4.2 while if $j = i + 1$ then $\{t_i^\alpha : \alpha \in F\}$ would contradict the assumption that $\mathbf{Q}$ is κ-well-ordered.

4.6. DEFINITION. $\mathrm{UF}\, B_X(\lambda, \kappa, \alpha)$ means that for every λ-X-barrier B of depth $\leqq \alpha$, there is $A \subseteq \lambda$, $|A| = \kappa$, such that $X\,\mathrm{Seq}_{<\omega}(A) \cap B \subseteq X\,\mathrm{Seq}_{\leqq n}(A)$ for some n.

4.7. LEMMA. (1) *If κ is weakly compact, $\alpha < \kappa$ then* $\mathrm{UF}\, B_X(\kappa, \kappa, \alpha)$ *holds* $(X = I, D)$.

(2) *If κ is Ramsey, then* $\mathrm{UF}\, B_I(\kappa, \kappa, \alpha)$ *for every α.*

PROOF. (1) By induction on α.

For $\alpha < \omega$, this is trivial. For $\alpha \geqq \omega$, let $\alpha = \delta + n$, δ a limit ordinal, n finite, and let B be a κ-X-barrier of depth α.

As κ is weakly compact, $\kappa \to (\kappa)^{n+1}_{|\alpha|}$, hence there is a set $A \subseteq \kappa$ of power κ, such that if $\eta \in X\,\mathrm{Seq}_{\leqq n+1}(A)$, $\mathrm{Dp}(\eta, B)$ depends only on $l(\eta)$ (and the order-relations between the $\eta(m)$ when $X = D$). Hence $S = \{\mathrm{Dp}(\eta, B) : \eta \in X\,\mathrm{Seq}_{\leqq n+1}(A)\}$ is finite. Now if $\eta \in X\,\mathrm{Seq}_m(A)$ then $\mathrm{Dp}(\eta, B) \leqq \delta + n - m$ (prove by induction on m) hence for $\eta \in X\,\mathrm{Seq}_{n+1}(A)$, $\mathrm{Dp}(\eta, B) < \delta$. Let $\delta^* = \mathrm{Max}(S \cap \delta)$, so for any $\eta \in X\,\mathrm{Seq}_{<\omega}(A)$, $\mathrm{Dp}(\eta, B) < \delta \Rightarrow \mathrm{Dp}(\eta, B) \leqq \delta^*$ (if $l(\eta) \leqq n + 1$ — obvious, otherwise $\mathrm{Dp}(\eta, B) \leqq \mathrm{Dp}(\eta \restriction (n+1), B) \leqq \delta^*$). So $\mathrm{Dp}(\langle\ \rangle, B \cap X\,\mathrm{Seq}_{<\omega}(A)) \leqq \delta^* + n + 1 < \alpha$, and we can apply the induction hypothesis.

(2) Obvious.

4.8. LEMMA. $\mathrm{UF}\, B_X(\lambda, \kappa, \alpha)$ *when $\lambda \geqq h(\kappa, \alpha)$, where h is defined by induction on α:*

(i) *for $\alpha < \omega$, $h(\kappa, \alpha) = \kappa$;*

(ii) *for $\alpha = \delta$, limit $h(\kappa, \alpha)$ is the first λ which is $\geqq h(\kappa, \beta)$ for every $\beta < \alpha$ and has cofinality $> \mathrm{cf}(\alpha)$;*

(iii) *for $\alpha = \delta + n$, $0 < n < \omega$, $\delta > 0$ limit, $h(\kappa, \alpha) = \beth_n(\Sigma_{\beta<\delta} h(\kappa, \beta))^+$ or even $\beth_{n-1}(2^{<|\Sigma h(\kappa,\beta)|})^+$.*

PROOF. We prove this by induction on α. W.l.o.g. $\lambda = h(\kappa, \alpha)$, $\mathrm{Dom}\, B = \lambda$.

Case I. $\alpha < \omega$

There is nothing to prove.

Case II. $\alpha = \delta$ *limit*

For every $i < \lambda$, $\mathrm{Dp}(\langle i \rangle, B) < \alpha$, so as $\mathrm{cf}\,\lambda = \mathrm{cf}\, h(\kappa, \alpha) > \mathrm{cf}(\alpha)$, for some $\beta < \alpha$, $A = \{i < \lambda : \mathrm{Dp}(\langle i \rangle, B) \leqq \beta\}$ has power λ. Now let $B' = B \cap X\,\mathrm{Seq}_{<\omega}(A)$, $\alpha' = \beta + 1$, and apply the induction hypothesis.

Case III. $\alpha = \delta + n$, $n > 0$, δ *limit*

We can easily prove by induction on k, that if $\eta \in X\,\mathrm{Seq}_k(\kappa)$, $\mathrm{Dp}(\eta, B) \leqq \delta + (n-k)$ and if $k > n$, $\mathrm{Dp}(\eta, B) < \delta$. Let $\delta = \bigcup_{i < \mathrm{cf}\,\delta} \alpha_i$, $\alpha_i < \delta$, $i < j \Rightarrow \alpha_i < \alpha_j$, $\lambda_i = h(\kappa, \alpha_i + n + 1)$ and we define a function G:

$$\mathrm{Dom}\, G = I\,\mathrm{Seq}_{n+1}(\kappa),$$

$$G(\eta) = \min\{i : \alpha_i \geqq \mathrm{Dp}(\eta', B) \text{ where } \eta' \in X\,\mathrm{Seq}_{n+1}(\kappa), \text{range}(\eta') \subseteq \text{range}(\eta)\}.$$

By the Erdos–Rado Theorem we know that $\lambda \rightarrow (\lambda_i)^{n+1}_{i<\mathrm{cf}\,\delta}$, so there are $i < \mathrm{cf}\,\delta$, $A \subseteq \lambda = \mathrm{Dom}\, B$, the order-type of A is λ_i and $G \restriction I\,\mathrm{Seq}_{n+1}(A)$ has the constant value i. By the definition of λ_i and the induction hypothesis we finish.

4.9. Lemma. *If* $\mathrm{UF}\,B_X(\lambda, \kappa, \alpha)$, $\mathbf{Q}$ *is* (κ, n)-X-bqo *for every* n, *then* $\mathbf{Q}$ *is* (λ, α)-X-bqo.

Proof. Trivial.

4.10. Lemma. *Suppose that for some* α, $\mathcal{P}_\alpha(\mathbf{Q})$ *is not* λ-D-*well-ordered. Then there are* $\gamma < \lambda^+$, *and* $t_i \in \mathcal{P}_\gamma(\mathbf{Q})$ $(i < 2^\lambda)$ *of hereditary power* $\leqq \lambda$ (*i.e.*, $|Tc(t_i)| \leqq \lambda$) *which are pairwise incomparable.*

Remark. If $\mathcal{P}_\alpha(\mathbf{Q})$ is not λ-D-well-ordered, then $\mathcal{P}_{\alpha+\gamma}(\mathbf{Q})$ is not $\beth_\gamma(\lambda)$-D-well-ordered at least when there is no strongly inaccessible κ, $\lambda < \kappa \leqq \gamma$.

Proof. Choose $s_i \in \mathcal{P}_\alpha(\mathbf{Q})$ $(i < \lambda)$ pairwise incomparable. Choose μ regular such that λ, $\mathbf{Q}$, α, s_i, $\mathcal{P}_{\alpha+1}(\mathbf{Q}) \in H(\mu)$ ($H(\mu)$ is the family of sets (in the universe of set theory), with transitive closure of power $< \mu$), and let N be an elementary submodel of $H(\mu)$, $\lambda + 1 \subseteq N$, of power λ to which $\langle s_i : i < \lambda \rangle$, $\mathbf{Q}$, $\mathcal{P}_{\alpha+1}(\mathbf{Q})$ belong. Let s'_i be s_i as interpreted in N (in other words, collapse the $\mathcal{P}$ hierarchy over $\mathbf{Q}$).

It is easy to check that if γ is the order-type of $N \cap \mu$ (which is $< \lambda^+$) then $s'_i \in \mathcal{P}_\gamma(\mathbf{Q})$ $(i < \lambda)$ and they are still pairwise incomparable (by absoluteness). Now let $S_\alpha \subseteq \lambda$ $(\alpha < 2^\lambda)$ be subsets of λ incomparable by inclusion, and $t_\alpha = \{s'_i : i \in S_\alpha\} \in \mathcal{P}_{\gamma+1}(\mathbf{Q})$ are as required.

§5. The trees

5.1. Definition. (1) $\mathcal{T}^0$ is the class of trees $\mathbf{T} = (T, \leqq)$ of height $\leqq \omega$ with a root $\mathrm{rt}(\mathbf{T}) = \mathrm{rt}_{\mathbf{T}}$ and T_n is the set of elements from level n, for $x \in T_{n-1}$,

$S_T(x) = \{y : x < y \in T_n\}$, so $S_T(x)$ is the set of immediate successors of x. We write sometimes $S(x)$ instead of $S_T(x)$.

(2) For each tree $\mathbf{T} = (T, \leqq)$ a depth function is defined:

(i) $\mathrm{Dp}_T(x) = \sup\{\mathrm{Dp}_T(y) + 1 : y \in S_T(x)\}$,

(ii) $\mathrm{Dp}_T(x) = \infty$ if not eventually defined by (i).

Let $\mathrm{Dp}(\mathbf{T}) = \mathrm{Dp}_T(\mathrm{rt}_T)$.

(3) $\mathcal{T}^0_{\leqq\alpha} = \{\mathbf{T} \in \mathcal{T}^0 : \mathrm{Dp}(\mathbf{T}) \leqq \alpha\}$; $\mathcal{T}^0_{<\infty} = \bigcup\{\mathcal{T}^0_{\leqq\alpha} : \alpha \text{ an ordinal}\}$.

(4) For $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}^0$, an embedding $f : \mathbf{T}_1 \to \mathbf{T}_2$ is a function preserving the order and the level and is not necessarily one to one.

5.2. Definition. (1) $\mathcal{T}^1$ is the class of models $M = (T, \leqq, <)$ such that $(T, \leqq) \in \mathcal{T}^0$, and $<$ is a partial order which is union of $<_x$ $(x \in T)$ where $<_x$ well orders $S(x)$.

(2) $\mathcal{T}^2$ is the class of models $M = (T, \leqq, E, <)$ such that $(T, \leqq) \in \mathcal{T}^0$, E is an equivalence relation, such that each equivalence class is included in some $S(x)$, and $<$ well orders each equivalence class, and $x < y \to xEy$.

(3) $\mathcal{T}^l(\mathbf{Q})$ is defined as $\mathcal{T}^l$ but we add to the model a function from the tree to $\mathbf{Q}$. We look at it as writing an element of $\mathbf{Q}$ on each node.

(4) An embedding for $\mathcal{T}^l$ is defined as in $\mathcal{T}^0$, but it has to preserve the additional relations (but not their negations). For $\mathcal{T}^l(\mathbf{Q})$ we have to demand $q(x) \leqq q(h(x))$, as usual.

(5) $\mathcal{T}^l_{\leqq\alpha}(\mathbf{Q})$, $\mathcal{T}^l_{<\infty}(\mathbf{Q})$ are defined as in 5.1(3).

Embeddability naturally quasi-orders $\mathcal{T}^l$, $\mathcal{T}^l(\mathbf{Q})$.

5.3. Main Theorem. *If $\lambda \geqq \aleph_0$, $\mathbf{Q}$ a $[\kappa;\lambda]$-bqo then $\mathcal{T}^2(\mathbf{Q})$ is $[\kappa;\lambda]$-bqo too (hence, also, $\mathcal{T}^0(\mathbf{Q})$, $\mathcal{T}^1(\mathbf{Q})$ are $[\kappa;\lambda]$-bqo).*

Proof. We first prove two claims and then return to the theorem.

Now for each ω-tree $\mathbf{T}$ and ordinal α, we defined the αth approximation $\mathbf{T}^\alpha$ to it:

the elements are $\{(s,\eta) : s \in T_n,\ l(\eta) = n+1,\ \alpha = \eta(0) > \eta(1) > \cdots > \eta(n)$ for some $n < \omega\}$, and

$(s_0, \eta_0) \leqq (s_1, \eta_1)$ iff $s_0 \leqq s_1$, $\eta_0 \leqq \eta_1$.

If $M = (\mathbf{T}, E, <, q)$ is a $\mathcal{T}^2(\mathbf{Q})$-tree we can similarly define its αth approximation $M^\alpha = (\mathbf{T}^\alpha, E^\alpha, <^\alpha, q^\alpha)$. We have to define $q^\alpha((s,\eta))$, $E^\alpha_{(s,\eta)}$, $<^\alpha_{(s,\eta)}$ for $(s,\eta) \in \mathbf{T}^\alpha$.

Let $q^\alpha((s,\eta)) = q(s)$. If $s \in \mathbf{T}_n$, $(s,\eta) \in \mathbf{T}^\alpha$, then $E^\alpha_{(s,\eta)} = \{\langle (t_1,\nu),(t_2,\nu)\rangle : t_1, t_2 \in S_T(s),\ t_1 E_s t_2,\ \nu = \eta\hat{\ }\langle i\rangle,\ i < \eta(l(\eta)-1)\}$, and $<^\alpha_{(s,\eta)} = \{\langle (t_1,\nu),(t_2,\nu)\rangle : t_1 <_s t_2,\ \nu = \eta\hat{\ }\langle i\rangle,\ \eta(l(\eta)-1) > i\}$.

5.4. CLAIM. (1) Every αth approximation to M has an embedding into it. Also if $\alpha \leqq \beta$, the αth approximation has an embedding into the βth approximation.

(2) If $M_l \in \mathcal{T}^2(\mathbf{Q})$ for $l = 0, 1$, $\lambda \geqq \|M_0\| + \|M_1\|$, λ infinite, and for every $\alpha < (2^\lambda)^+$, there is an embedding of M_0^α into M_1 *then* there is an embedding of M_0 into M_1.

(3) $\mathrm{Dp}(M^\alpha)$, $\leqq \alpha$.

(4) If $M_l \in \mathcal{T}^2(\mathbf{Q})$, $\lambda \geqq \|M_0\| + \|M_1\|$, λ infinite, $\alpha = (2^\lambda)^+$ and $M_0^\alpha \leqq M_1^\alpha$ then $M_0 \leqq M_1$.

PROOF OF 5.4. (1) Just map (s, η) to s. The second phrase is easy too.

(2) Let $g_\alpha : M_0^\alpha \to M_1$ be a $\mathcal{T}^2(\mathbf{Q})$ embedding. We now define by induction on n, for every $s \in (M_0)_n$ = the nth level a set $A_s \subseteq (2^\lambda)^+$, $|A_s| = (2^\lambda)^+$, and for each $\zeta \in A_s$, an ordinal $\alpha^s(\zeta)$, $\zeta < \alpha^s(\zeta) < (2^\lambda)^+$ and a decreasing sequence η_ζ^s such that $l(\eta_\zeta^s) = n + 1$, $\eta_\zeta^s(0) = \alpha^s(\zeta)$, $\eta_\zeta^s(n) \geqq \zeta$ and a function $h_s : \{rt_{\mathbf{T}_0}\} \cup \bigcup_{t <_s} S_{\mathbf{T}_0}(t) \to M_1$ such that for every $a \in \mathrm{Dom}(h_s)$, $a \in (M_0)_{m-1}$, $h_s(a) = g_{\alpha^s(\zeta)}((a, \eta_\zeta^s \restriction m))$ for all $\zeta \in A_s$.

We assure also that the functions h_s are consistent. It is easy to define and show $\bigcup_{s \in \mathbf{T}_0} h_s$ is the required embedding.

(3) Easy — prove by induction on γ that $s \in M_n$, $\eta(n) \leqq \gamma$, $(s, \eta) \in M^\alpha$ implies $\mathrm{Dp}((s, \eta)) \leqq \gamma$.

(4) By (1) and (2).

5.5. THE bqo CRITERION LEMMA. The following is a sufficient condition on a quasi-order $\mathbf{Q}$, for being $[\kappa; \lambda]$-X-bqo, when $\lambda = 1$ or $\lambda \geqq \aleph_0$. Let $S_{\mathbf{Q}} = \{(q_1, q_2) : q_1 \in \mathbf{Q},\ q_2 \in \mathbf{Q}$ but not $q_1 \leqq q_2\}$.

THE CRITERION. There is a (rank) function rk from $\mathbf{Q}$ to ordinals (or any well-ordered class, or even well-founded one), a two place function s from $S_{\mathbf{Q}}$ to $\mathbf{Q}$ and a function F^* from $S_{\mathbf{Q}}$ to λ such that:

(a) for no q_1, q_2, q_3, $(q_1, q_2) \in S_{\mathbf{Q}}$, $(q_2, q_3) \in S_{\mathbf{Q}}$; $q_1 \neq s(q_1, q_2) \leqq s(q_2, q_3) \neq q_2$ and $F^*(q_1, q_2) = F^*(q_2, q_3)$;

(b) if $t \in \mathbf{Q}$ is not with minimal rank then $s(t, q) \neq t$ and $\mathrm{rk}[s(t, q)] < \mathrm{rk}[t]$;

(c) the set of members of minimal rank of $\mathbf{Q}$, i.e., $\mathbf{Q}_m = \{q \in Q : \text{for no } t,\ \mathrm{rk}(t) < \mathrm{rk}(q)\}$, is $[\kappa; \lambda]$-X-bqo;

(d) if $\lambda < \aleph_0$ then $s(t, q) \leqq t$;

(e) if $\lambda < \aleph_0$, $(q_1, q_2) \in S_{\mathbf{Q}}$, $\mathrm{rk}(q_2)$ is minimal but not $\mathrm{rk}(q_1)$ then not $s(q_1, q_2) \leqq q_2$.

REMARK. For $\lambda = 1$, $X = I$ this is essentially the forerunner criterion of Nash-Williams, but his formulation involves barriers and he used a stronger definition of a barrier, so also the domain of the barrier is changed, and the notion "warily forerun" is involved. In fact we move more of the proof to the criterion.

PROOF OF 5.5. Suppose $\mathbf{Q}$ is not $[\kappa;\lambda]$-X-bqo and we shall get a contradiction. So there is a κ-X-barrier B, $q_\eta \in \mathbf{Q}$ for $\eta \in B$ and a function $F: B \to \lambda$, such that for no $\eta, \nu \in B$, $\eta \mathbf{R}^1_X \nu$, $F(\eta) = F(\nu)$, and $q_\eta \leqq q_\nu$.

We now define by induction on n, B_n, $C_n \subseteq B_n$, $q^n_\eta \in \mathbf{Q}(\eta \in B_n)$, and F_n such that B_n is a κ-X-barrier, $\operatorname{Dom} B_n = \operatorname{Dom} B$, F_n a function, $B_n = \operatorname{Dom} F_n$, $\lambda \geqq |\operatorname{Range} F_n|$ such that for no $\eta, \nu \in B_n$, $\eta \mathbf{R}^1_X \nu$, $F_n(\eta) = F_n(\nu)$, and $q^n_\eta \leqq q^n_\nu$, and $C_n = \{\eta \in B_n : q^n_\eta \text{ is of minimal rank}\}$.

For $n = 0$. $B_n = B$, $q^n_\eta = q_\eta$, F_n is $F_n(\eta) = \langle F(\eta), 0\rangle$;

$$C_n = \{\eta \in B_n : q^n_\eta \text{ has minimal rank}\}.$$

For $n+1$. Let $B_{n+1} = B^0_{n+1} \cup C_n$ where

$$B^0_{n+1} = \{\eta \cup^* \nu : \eta \in B_n,\ \nu \in B_n,\ \eta \mathbf{R}^1_X \nu,\ \eta \notin C_n\}.$$

Let $C_{n+1} = C_n \cup \{\eta \cup^* \nu : \eta \cup^* \nu \in B^0_{n+1}, \eta \mathbf{R}^1_X \nu$ and $s(q^n_\eta, q^n_\nu)$ is of minimal rank or $F_n(\eta) \neq F_n(\nu)\}$. Choose $q^* \in \mathbf{Q}_m$ arbitrarily. Let us define q^{n+1}_σ $(\sigma \in B_{n+1})$

$$q^{n+1}_\sigma = \begin{cases} s(q^n_\eta, q^n_\nu) & \text{if } \sigma = \eta \cup^* \nu \in B^0_{n+1},\quad \eta \mathbf{R}^1_X \nu,\quad F_n(\eta) = F_n(\nu),\\ q^* & \text{if } \sigma = \eta \cup^* \nu \in B^0_{n+1},\quad \eta \mathbf{R}^1_X \nu,\quad F_n(\eta) \neq F_n(\nu),\\ q^n_\sigma & \text{if } \sigma \in C_n.\end{cases}$$

We now define F_{n+1}:

$$F_{n+1}(\sigma) = \begin{cases} F_n(\eta) & \text{if } \lambda < \aleph_0,\quad \sigma = \eta \cup^* \nu \in B^0_{n+1},\quad \eta \mathbf{R}^1_X \nu,\\ \langle u, F_n(\eta), F_n(\nu), n+1\rangle & \text{if } \lambda \geqq \aleph_0,\quad \sigma = \eta \cup^* \nu \in B^0_{n+1},\quad \eta \mathbf{R}^1_X \nu,\\ & \quad \text{where } u = F^*(q^n_\eta, q^n_\nu) \text{ if } q^n_\eta \not\leqq q^n_\nu\\ & \quad \text{and } u = \infty \text{ otherwise},\\ F_n(\sigma) & \text{if } \sigma \in C_n.\end{cases}$$

FACT A. The induction hypothesis is satisfied.

For $n = 0$, this is an assumption. For $n+1$, let us check

(A1) B_{n+1} is an X-barrier with domain $\operatorname{Dom} B$.

Let $\eta \in X\operatorname{Seq}_\omega(\operatorname{Dom} B)$, and we shall show that for some l, $\eta \upharpoonright l \in B_{n+1}$. For some k, $\eta \upharpoonright k \in B_n$, and for some m, $\eta^- \upharpoonright m \in B_n$. If $\eta \upharpoonright k \in C_n$ then $\eta \upharpoonright k \in B_{n+1}$, otherwise $(\eta \restriction k) \cup^* (\eta^- \restriction m) \lessdot \eta$ is in B^0_{n+1} hence in B_{n+1}. Now the other conditions for "B_{n+1} is an X-barrier" are trivial in our case, and $\operatorname{Dom} B_{n+1} = \operatorname{Dom} B$, obvious by the above.

(A2) $\qquad C_n = \{\eta \in B_n : q^n_\eta \text{ minimal}\}, \quad |\operatorname{Range} F_n| \leqq \lambda.$

The proof is easy.

(A3) Suppose $\sigma_1, \sigma_2 \in B_{n+1}$, $F_{n+1}(\sigma_1) = F_{n+1}(\sigma_2)$, $\sigma_1 \mathbf{R}^1_X \sigma_2$. We shall prove that not $q^{n+1}_{\sigma_1} \leqq q^{n+1}_{\sigma_2}$.

We check by cases:

Case (i). $\sigma_1, \sigma_2 \in C_n$

Then $\sigma_1, \sigma_2 \in B_n$, $q^{n+1}_{\sigma_1} = q^n_{\sigma_1}$, $q^{n+1}_{\sigma_2} = q^n_{\sigma_2}$, $F_{n+1}(\sigma_1) = F_n(\sigma_1)$, $F_{n+1}(\sigma_2) = F_n(\sigma_2)$, so use the induction hypothesis.

Case (ii). $\sigma_1, \sigma_2 \in B^0_{n+1}$

Then let for $l = 1,2$: $\sigma_l = \eta_l \cup^* \nu_l$, $\eta_l \in B_n$, $\nu_l \in B_n$, $\eta_l \notin C_n$.

As $\sigma_1^- \leqq \sigma_2$, clearly $\nu_1 \leqq \sigma_2$ hence ν_1, η_2 are comparable but both are in the X-barrier B_n, so $\nu_1 = \eta_2$. As $F_{n+1}(\sigma_1) = F_{n+1}(\sigma_2)$, necessarily by F_{n+1}'s definition, $F_n(\eta_1) = F_n(\eta_2)$ and $F_n(\nu_1) = F_n(\nu_2)$, so by the induction hypothesis not $q^n_{\eta_1} \leqq q^n_{\nu_1}$ nor $q^n_{\eta_2} \leqq q^n_{\nu_2}$. So $(q^n_{\eta_l}, q^n_{\nu_l}) \in S_Q$. Also $F^*(q^n_{\eta_1}, q^n_{\nu_1}) = F^*(q^n_{\eta_2}, q^n_{\nu_2})$: we have already proved they are defined. They are equal; if $\lambda \geqq \aleph_0$, by the definition of F_{n+1}, if $\lambda < \aleph_0$ as $|\operatorname{Range} F^*| = 1$.

By the induction hypothesis, if $\eta \in B_n$, q^n_η is of minimal rank, then $\eta \in C_n$. Hence, as $\eta_l \notin C_n$, $q^n_{\eta_1}$ and $q^n_{\eta_2}$ are not of minimal rank.

Now let $q_1 = q^n_{\eta_1}$, $q_2 = q^n_{\nu_1} = q^n_{\eta_2}$, $q_3 = q^n_{\nu_2}$, so we have proved that q_1, q_2 are not of minimal rank, hence (by part (b) of the criterion), $s(q_1, q_2) \neq q_1$, $s(q_2, q_3) \neq q_2$; and also not $q_1 \leqq q_2$ nor $q_2 \leqq q_3$ (by induction hypothesis on n) and $F^*(q_1, q_2) = F^*(q_2, q_3)$. So by (a) of the criterion, not $s(q_1, q_2) \leqq s(q_2, q_3)$, that is not $s(q^n_{\eta_1}, q^n_{\nu_1}) \leqq s(q^n_{\eta_2}, q^n_{\nu_2})$. So by the definition of q^{n+1}_η this means not $q^{n+1}_{\sigma_1} \leqq q^{n+1}_{\sigma_2}$, so we finish Case (ii).

Case (iii). $\sigma_1 \in B^0_{n+1}$, $\sigma_2 \in C_n$ and $\sigma_2 \in B_n$. So $\sigma_1 = \eta_1 \dot{\cup}^* \nu_1$ as in (ii), $\sigma_2 \in B_n$.

So as $\sigma_1^- \leqq \sigma_2$, now $\nu_1 \leqq \sigma_2 \in B_n$ and $\nu_1 = \sigma_2$. If $\lambda \geqq \aleph_0$, by the definition of F_n, the last coordinate of $F_{n+1}(\sigma_1)$ is $n+1$, while for $F_{n+1}(\sigma_2)$ this is not the case, contradicting the assumption that $F_{n+1}(\sigma_1) = F_{n+1}(\sigma_2)$.

So assume $\lambda < \aleph_0$.

Now F_n, F^* are constant, hence by the induction hypothesis not $q^n_{\eta_1} \leqq q^n_{\nu_1}$ and as $\eta_1 \notin C_n$, $q^n_{\eta_1}$ is not minimal, but $\sigma_2 = \nu_1 \in C_n$, hence (we can prove by induction on n) $q^n_{\nu_1}$ has minimal rank. So by part (e) not $s(q^n_{\eta_1}, q^n_{\nu_1}) \leqq q^n_{\nu_1}$ which means $q^{n+1}_{\sigma_1} \not\leqq q^{n+1}_{\sigma_2}$ as required.

Case (iv). $\sigma_1 \in C_n$, $\sigma_2 \in B^0_{n+1}$

So let $\sigma_2 = \eta_2 \cup^* \nu_2$, $\eta_2 \in B_n$, $\nu_2 \in B_n$, $\eta_2 \mathbf{R}^1_X \nu_2$, $\eta_2 \not\in C_n$. As in case (iii) we know $\lambda < \aleph_0$, $F_n(\eta_2) = F_{n+1}(\sigma_2) = F_{n+1}(\sigma_1)$. So $q^{n+1}_{\sigma_1} = q^n_{\sigma_1}$, not $q^n_{\sigma_1} \leqq q^n_{\eta_2}$. Now if $q^n_{\eta_2} \geqq q^{n+1}_{\sigma_2}$, our conclusion is trivial. As $F_n(\eta_2) = F_n(\nu_2)$ by the induction hypothesis not $q^n_{\eta_2} \leqq q^n_{\nu_2}$, so by part (d) of the criterion $q^n_{\eta_2} \geqq s(q^n_{\eta_2}, q^n_{\nu_2}) = q^{n+1}_{\sigma_2}$, so we finish.

We thus finish the proof of (A3) hence of Fact A. Now let $B' = \bigcup_{n<\omega} C_n$, $q'_\eta = q^n_\eta$ for $\eta \in C_n - \bigcup_{l<n} C_l$, and $F'(\eta) = F_n(\eta)$ for $\eta \in C_n - \bigcup_{l<n} C_l$. We shall prove that B', q'_η, F' exemplify $\mathbf{Q}_m$ is not $[\kappa;\lambda]$-X-bqo (i.e., B' is a κ-X-barrier and $\eta, \nu \in B'$, $\eta \mathbf{R}^1_X \nu$, $F'(\eta) = F'(\nu)$ implies not $q'_\eta \leqq q'_\nu$), thus getting the required contradiction.

FACT B. If $\eta \in C_n$ then for every $m \geqq n$, $\eta \in C_m \subseteq B_m$, $q^m_\eta = q^n_\eta$, $F_m(\eta) = F_n(\eta)$.

PROOF OF FACT B. Check in the definition of F_n, C_n.

FACT C. If $\eta, \nu \in B'$, $\eta \mathbf{R}^1_X \nu$, $F'(\eta) = F'(\nu)$ then not $q'_\eta \leqq q'_\nu$.

PROOF. By Fact B, for every large enough n, $F'(\eta) = F_n(\eta)$, $F'(\nu) = F_n(\nu)$ and $\eta, \nu \in C_n \subseteq B_n$; and use Fact A.

FACT D. If $\eta \in X\,\mathrm{Seq}_\omega(\mathrm{Dom}\, B)$ then for some n, $\eta \restriction n \in B'$.

For every n, as B_n is a κ-X-barrier, $\mathrm{Dom}\, B_n = \mathrm{Dom}\, B$ (by Fact A) there is $k(n) < \omega$, $\eta \restriction k(n) \in B_n$. If for some n, $\eta \restriction k(n) \in C_n$ then by the definition of B', $\eta \restriction k(n) \in C_n \subseteq B'$, so we finish. Suppose $\eta_n = \eta \restriction k(n) \in B_n - C_n$ for every n. For each n, there is $m(n) < \omega$ such that $\nu_n = (\eta^-) \restriction m(n) \in B_n$. As B_n is an X-barrier $\eta^-_n \leqq \nu_n$ and even $\eta_n \mathbf{R}^1_X \nu_n$, so by the definition of B^0_{n+1}, $\eta_n \cup^* \nu_n \in B^0_{n+1} \subseteq B_{n+1}$, so necessarily $\eta_n \cup^* \nu_n = \eta \restriction (m(n)+1) = \eta_{n+1}$. As $\eta_{n+1} \not\in C_{n+1}$, clearly $F_n(\eta_n) = F_n(\nu_n)$, hence $q^{n+1}_{\eta_{n+1}} = s(q^n_{\eta_n}, q^n_{\nu_n})$. As $\eta_n \not\in C_n$, $q^n_{\eta_n}$ is not of minimal rank. Therefore by part (b) of the criterion $\mathrm{rk}(q^{n+1}_{\eta_{n+1}}) < \mathrm{rk}(q^n_{\eta_n})$. Since this holds for all $n < \omega$, this is a contradiction to the well-foundedness of the range of the rank function.

The other requirements for an X-barrier are easy to verify. This completes the proof of 5.5.

CONTINUATION OF THE PROOF OF 5.3. By Claim 5.4 (3), (4) it suffices to prove that $\mathcal{T}^2_{\leqq\alpha}(\mathbf{Q})$ is $[\kappa;\lambda]$-X-bqo for every α. We define a rank function from $\mathcal{T}^2_{\leqq\alpha}(\mathbf{Q})$

to triples of ordinals, ordered lexicographically. So let $M = (T, \leqq, E, <, q)$, and we define $\mathrm{rk}(M) = \langle \alpha_1(M), \alpha_2(M), \alpha_3(M)\rangle$ where $\alpha_1(M) = \mathrm{Dp}_M(x_{k-1}(M))$, where $k = k(M) = \min\{k' : |T_{k'}| \neq 1\}$, $x_l(M)$ is the unique $x \in T_l$ for $l < k$,

$$\alpha_2(M) = \begin{cases} 0, & T_k = \varnothing, \\ 1, & T_k \text{ is an } E\text{-equivalence class}, \\ 2, & \text{otherwise}, \end{cases}$$

$\alpha_3(M)$ is the order type of $(T_k, <)$ when it is well-ordered, and zero otherwise.

So $\mathbf{Q}_m$, the set of $M \in \mathbf{Q}$ with minimal rank, is just the set of M with $T_{k(M)} = \varnothing$. Now suppose $M, N \in \mathcal{T}^2_{\leqq\alpha}(\mathbf{Q})$ but not $M \leqq N$. We shall define $s(M,N)$ and $F^*(M,N)$ so that we can apply the criterion of 5.5.

Case I. $k(M) = k(N)$, $\alpha_2(M) = 2$

Checking the definitions it is clear that there is a set $A \subseteq M$, such that

(a) $A \cap M_{k(M)}$ is exactly one E-equivalence class, $x_l(M) \in A$ for $l < k$, and $x < y \in M \wedge x \in A \cap M_{k(M)} \rightarrow y \in A$,

(b) not $M \restriction A \leqq N$.

Now we let $s(M,N) = M \restriction A$, $F^*(M,N) = 1$.

Case II. $k(M) = k(N)$, $\alpha_2(M) = 1 = \alpha_2(N)$

Let $M_{k(M)} = \{a^0_i : i < \zeta_0\}$, where for $i < j$, $a^0_i <^M a^0_j$, and $N_{k(N)} = \{a^1_i : i < \zeta_1\}$ where for $i < j$, $a^1_i <^N a^1_j$. Let $\zeta_l = \beta_l + n_l$, where $n_l < \omega$, β_l is zero or limit. Let for $l = 0[1]$, A^l_i be the set of elements of M [of N] which are comparable with a^l_i.

By an assumption, not $M \leqq N$, hence there is no monotonic $f : \zeta_0 \rightarrow \zeta_1$ such that $M \restriction A^0_i \leqq N \restriction A^1_{f(i)}$ for $i < \zeta_0$. Hence not $M \restriction \bigcup\{A^0_i : i < \beta_0\} \leqq N \restriction \bigcup\{A^1_i : i < \beta_1\}$, or for some $l < n_0$, not $M \restriction A^0_{\beta_0+l} \leqq N \restriction A^1_{\beta_1+l}$. If the first thing occurs then basically† not $M \restriction \bigcup\{A^0_i : i < \beta_0\} \leqq N$ and we define $\gamma(i) < \beta_i$ by induction on $i < \beta_0$, as the first $\gamma < \beta_1$ such that $\gamma > \gamma(j)$ for $j < i$ and $M \restriction A^0_i \leqq N \restriction A^1_{\gamma(i)}$. We cannot succeed to define $\gamma(i)$ for every i, so let i^* be the first i for which $\gamma(i)$ is not defined.

We let $s(M,N) = M \restriction \bigcup\{A^0_i : i \leqq i^*\}$, $F^*(M,N) = 2$.

If the first thing does not occur, $l < n_0$ is minimal such that $M \restriction A^0_{\beta_0+l} \leqq N \restriction A^1_{\beta_1+l}$ fails (maybe $\beta_0 + l \geqq \zeta_1$) then we let

$$F^*(M,N) = \langle 3, l\rangle, \qquad s(M,N) = M \restriction A^0_{\beta_0+l}.$$

Case III. $k(M) \neq k(N)$, or $k(M) = k(N)$, $\alpha_2(M) = 1 \neq \alpha_2(N)$

We let $F^*(M,N) = \langle 4, k(M), k(N), \alpha_2(M), \alpha_2(N)\rangle$; $s(M,N)$ is a tree with a single element.

† The point is that we are interested in the case $F^*(M,N) = F^*(N,N')$.

Case IV. $k(M)=k(N)$, $\alpha_2(M)=0$

We let $s(M,N)=M$, $F^*(M,N)=5$.

Now it is easy to check that (a), (b), (c) of the criterion are satisfied, so we finish. (For (c) we have to prove that $\{M:\alpha_2(M)=0\}$ is $[\kappa;\lambda]$-bqo.

5.6. THEOREM. *The following are equivalent*:

(a) $\mathbf{Q}\times(\omega,\leqq)$ *is* κ-X-bqo,

(b) $\mathcal{P}^{**}_\alpha(\mathbf{Q})$ *is* κ-X-bqo *for every* α (see Definition 1.3(6)),

(c) $\mathcal{P}^{**}_\omega(\mathbf{Q})$ *is* κ-X-bqo,

(d) *for every* κ-X-*barrier* B, *and* $q_\eta\in\mathbf{Q}$ $(\eta\in B)$ *for some* $\eta,\nu\in B$, $\eta\mathbf{R}^1_X\nu$, $q_\eta\leqq q_\nu$ *and* $\eta^-\neq\nu$.

PROOF. *not* $(a)\Rightarrow$ *not* (c). Define $q^{(n)}\in\mathcal{P}^{**}_n(\mathbf{Q})$ by induction on n for $q\in\mathbf{Q}: q^{(0)}=q$, $q^{(n+1)}=\{q^{(n)}\}$. Now we assume $\mathbf{Q}\times(\omega,\leqq)$ is not κ-X-bqo, so there is a κ-X-barrier B, and $(q_\eta,k_\eta)\in\mathbf{Q}\times(\omega,\leqq)$, such that for no $\eta\mathbf{R}^1_X\nu$, $\eta\in B$, $\nu\in B$, and $(q_\eta,k_\eta)\leqq(q_\nu,k_\nu)$. Now define for $\eta\in B$, $t_\eta=q^{(k_\eta)}_\eta\in\mathcal{P}^{**}_\omega(\mathbf{Q})$. Now $q^{(n)}\leqq t^{(m)}$ iff $q\leqq t$, $n\leqq m$, so we finish.

$(b)\Rightarrow(c)$. Trivial.

$(a)\Rightarrow(d)$. As $\langle(q_\eta,l(\eta)):\eta\in B\rangle$ does not show $\mathbf{Q}\times(\omega,\leqq)$ is not κ-X-bqo, for some $\eta\in B$, $\nu\in B$, $\eta\mathbf{R}^1_X\nu$, and $(q_\eta,l(\eta))\leqq(q_\nu,l(\nu))$. So $q_\eta\leqq q_\nu$ and $l(\eta)\leqq l(\nu)$, hence $\eta^-\neq\nu$, so we finish.

$(d)\Rightarrow(b)$. Repeat the proof of 1.8

REMARK. Clearly $[\kappa;2]$-X-bqo implies (d) from the theorem (use $F:B\to 2$, $F(\eta)\equiv l(\eta)\bmod 2$). Also for $\kappa=\aleph_0$, $X=I$, (a), $\cdots$, (d) are equivalent to X-bqo.

5.7. THEOREM. *The well-ordering number of* $\mathcal{T}^0$ *ordered by one-to-one embeddability is the first beautiful cardinal* $\kappa_0>\aleph_0$.

PROOF. By 5.3, $\mathcal{T}^0$ is κ_0-I-bqo, even under one-to-one embedding; now let $\lambda<\kappa_0$, and we prove that it is not λ-narrow. Let $\mathbf{Q}=(\omega,=)$ so it is not $\aleph_0$-narrow, hence by 2.5 for some α $\mathcal{P}_\alpha(\mathbf{Q})$ is not λ-narrow, so let $t_i\in\mathcal{P}_\alpha(\mathbf{Q})$ $(i<\lambda)$ be pairwise incomparable. Let S_i $(i\leqq\omega)$ be infinite, pairwise disjoint subsets of ω, $\bigcup S_i=\{3n+1:n<\omega\}$. For each $i<\lambda$ we shall define a tree $\mathbf{T}_i\in\mathcal{T}^0$. The elements of T_i are the sequences $\langle s_0,\cdots,s_m\rangle$ such that:

(a) $s_0=t_i$,

(b) if $s_l\in Tc(t_i)$, $s_l\notin\mathbf{Q}$, $l<m$ and $l\in S_\omega$ *then* $s_{l+1}\in s_l$,

(c) if $s_l\in Tc(t_i)$, $s_l\notin\mathbf{Q}$, $l<m$ and $l\notin S_\omega$ *then* $s_{l+1}=s_l$,

(d) if $s_l\in\mathbf{Q}$ (so s_l is a natural number), $l<m$ and $l\notin S_{s_l}$ *then* $s_{l+1}=s_l$,

(e) if $s_l\in\mathbf{Q}$, $l\in S_{s_l}$ and $l<m$ *then* $s_{l+1}\in\{\omega,\omega+1\}$, $s_{l+2}\in\{\omega+2,\omega+3\}$,

(f) if $s_l\geqq\omega+2$, then $l=m$ (so s_{l+1} is not defined).

The order $\leqq$ on T_i is an initial segment.

We leave the reader to check that $\mathbf{T}_i = (T_i, \leqq)$ $(i < \lambda)$ are pairwise incomparable.

REMARK. (1) $\mathcal{T}^0$ has well-ordering number $\aleph_0$, as $\mathbf{T}_1 \leqq \mathbf{T}_2$ iff the depth of $\mathbf{T}_1$ is $\leqq$ the depth of $\mathbf{T}_2$.

(2) The choice of $\mathbf{T}_i$ is such that they will be suitable for 5.8 too. The point is that we should be able to easily reconstruct from the tree what form each s_l has, hence t_i.

5.8. THEOREM. *Suppose* $\mathbf{Q}$ *is non-trivial in the sense that for some* $q_1, q_2 \in \mathbf{Q}$, $q_1 \not\leqq q_2$

Then the well-ordering number of $\mathcal{T}^0(\mathbf{Q})$ *and even* $\mathcal{T}^0_{<\infty}(\mathbf{Q})$ *is the first* κ *for which* $\mathbf{Q}$ *is* $[\kappa;\aleph_0]$-bqo (see 2.13(2)).

PROOF. If $\mathbf{Q}$ is $[\kappa;\aleph_0]$-bqo, then by 5.3, $\mathcal{T}^0(\mathbf{Q})$ is $[\kappa;\aleph_0]$-bqo hence κ-well-ordered. Now suppose $\mathbf{Q}$ is not $[\lambda;\aleph_0]$-bqo, then by 2.11, $\mathbf{Q}\times(\omega, =)$ is not λ-D-bqo, then for some α, $\mathcal{P}_\alpha(\mathbf{Q}\times(\omega, =))$ is not λ-D-well-ordered, so there are $t_i \in \mathcal{P}_\alpha(\mathbf{Q}\times(\omega, =))$ $(i<\lambda)$ pairwise incomparable. We define $\mathbf{T}_i$ as in the proof of 5.7 replacing "$s_l \in \mathbf{Q}$" by "$s_l = (q_l, n_l) \in \mathbf{Q}\times(\omega, =)$" and S_{s_l} by S_{n_l}. We then define $f_i : \mathbf{T}_i \to \mathbf{Q}$ by:

(a) if $\langle s_0, \cdots, s_m\rangle \in \mathbf{T}_i$, $s_m = (q, n) \in \mathbf{Q}\times(\omega, =)$, $m \in S_n$ then $f_i(\langle s_0, \cdots, s_m\rangle) = q$,

(b) if $\bar{s} = \langle s_0, \cdots, s_m\rangle \in \mathbf{T}_i$, $s_m = \omega$ or $s_m = \omega + 3$ then $f_i(\bar{s}) = q_1$,

(c) in the other cases $f_i(\bar{s})$ is q_2.

We leave the rest to the reader.

§6. Unions of few scattered orders

6.1. DEFINITION. (1) A (linearly) ordered set $\mathbf{I} = (I, \leqq)$ is *scattered* if we cannot embed the rationals into it. Equivalently, by a theorem of Hausdorff, they are generated from $\{1\}$, by well-ordered and inversely well-ordered sums.

(2) $\mathfrak{M}_\lambda$ is the class of linearly ordered sets $\mathbf{I}$, which can be partitioned into $\leqq \aleph_0$ scattered orders. $\mathfrak{M}_\lambda$ is naturally quasi-ordered by embeddability.

6.2. DEFINITION. Let $\eta, \nu \in \mathrm{Seq}_{<\omega}(\alpha)$. We say that $\eta <^x \nu$ *iff* for some k, $\eta \restriction k = \nu \restriction k$, $\eta \neq \nu$ and

$$[k \text{ even}, \eta(k) < \nu(k)] \quad \text{or}$$

$$[k \text{ even}, l(\eta) = k] \quad \text{or}$$

$$[k \text{ odd}, \eta(k) > \nu(k)] \quad \text{or}$$
$$[k \text{ odd}, l(\nu) = k].$$

The relation $<^x$ linearly orders $\mathrm{Seq}_{<\omega}(\alpha)$.

6.3. THEOREM. (1) Every scattered order is isomorphic to some $\langle A, \leqq^x \rangle$, $A \subseteq \mathrm{Seq}_{<\omega}(\alpha)$.

(2) *Every* $\mathbf{I} \in \mathfrak{M}_{\aleph_0}$ *is isomorphic to* $\langle A, \leqq^x \rangle$, $A \subseteq \mathrm{Seq}_{<\omega}(\alpha)$.

PROOF. See Laver [10].

6.4. DEFINITION. (1) A triple $P = (I, F_1, F_2)$ is a λ*-representation* if

(a) I is a set of sequences of ordinals closed under initial segments,

(b) $F_1(\eta) = 0$ iff for no i, $\eta^\wedge\langle i \rangle \in I$,

(c) if $F_1(\eta) = 1$ then $\{i : \eta^\wedge\langle i \rangle \in I\}$ is some ordinal $\alpha_\eta \leqq \lambda$,

(d) $\mathrm{Dom}\, F_2 = F_1^{-1}(\{1\})$, $F_2(\eta)$ is an ordered set with universe α_η,

(e) $\mathrm{Range}\, F_1 = \{0, 1, 2, 3\}$.

(2) P is called *standard* if I is standard, which means: $I \subseteq \mathrm{Seq}_{<\omega}(\alpha)$ is closed under initial segments and $\eta^\wedge\langle i \rangle \in I \wedge j < i \Rightarrow \eta^\wedge\langle j \rangle \in I$.

(3) The order $J[P] = \mathbf{J}^P = (J^P, <^P)$ which P represents is defined by: $J^P = F_1^{-1}(\{0\})$;

$$\begin{aligned} \eta <^P \nu \textit{ iff } & F_1(\eta) = F_1(\nu) = 0, \text{ and for some } k \\ & \eta \restriction k = \nu \restriction k,\ \nu \restriction (k+1) \neq \eta \restriction (k+1) \\ & \text{and } F_1(\eta \restriction k) = 1 \Rightarrow F_2(\eta \restriction k) \vDash \eta(k) < \nu(k) \\ & \text{and } F_1(\eta \restriction k) = 2 \Rightarrow \eta(k) < \nu(k) \\ & \text{and } F_1(\eta \restriction k) = 3 \Rightarrow \eta(k) > \nu(k). \end{aligned}$$

6.5. CLAIM. Suppose P_1, P_2 are λ-representations, $f : P_1 \to P_2$ (i.e., $f : I^{P_1} \to I^{P_2}$) is a one-to-one function, it preserves the level, F_1, the order $\lessdot$ and if $f(\eta^\wedge\langle i_k \rangle) = \nu^\wedge\langle j_l \rangle$ $(l = 1, 2)$ then

(a) $F_1^{P_1}(\eta) = 2, 3 \Rightarrow [i_1 < i_2 \equiv j_1 < j_2]$;

(b) $F_1^{P_1}(\eta) = 1 \Rightarrow [F_2^{P_1}(\eta) \vDash i_1 < i_2 \Leftrightarrow F_2^{P_2}(\nu) \vDash j_1 < j_2]$.

Then $J[P_1]$ is embeddable in $J[P_2]$.

PROOF. Trivial.

6.6. THEOREM. *Every member* $\mathbf{I}$ *of* $\mathfrak{M}_\lambda$ *has a standard* λ*-representation.*

PROOF. We prove this by induction on the power of the order. If $|\mathbf{I}| \leqq \lambda$, this is trivial, so suppose $|\mathbf{I}| = \mu > \lambda$. As $\mathbf{I} \in \mathfrak{M}_\lambda$, there are $A_i \subseteq \mathrm{Seq}_{<n(i)}(\mu)$, $\mathbf{I} = \bigcup_{i<\lambda} \mathbf{I}_i$, $\mathbf{I}_i \cong (A_i, \leqq^x)$ (see 6.3), and let $g_i : A_i \to \mathbf{I}_i$ be the isomorphism. W.l.o.g.

$A_i = \mathrm{Seq}_{<n(i)}(\mu)$, since any partially ordered set can be linearly ordered by a relation extending the partial order, and if we have a λ-representation of an order we can easily obtain a λ-representation of any subset with the induced order.

We get $\mathbf{I}^c$ from $\mathbf{I}$ by inserting in each Dedekind cut two elements (small and big). Let $I_i^0 = \{a \in I_i : l(g_i^{-1}(a)) < n(i) - 1\}$, $I_i^1 = \{a \in I_i : g_i^{-1}(a) = \eta\hat{\ }\langle\delta\rangle$ for some η and limit $\delta\}$. We define $h_i^l : I_i^l \to I^c$, $l = 0, 1$, by $h_i^0(g_i(\eta)) = \lim_{\alpha\to\mu} g_i(\eta\hat{\ }\langle\alpha\rangle)$ and $h_i^1(g_i(\eta\hat{\ }\langle\delta\rangle)) = \lim_{\alpha\to\delta} g_i(\eta\hat{\ }\langle\alpha\rangle)$. Let $I^* = I \cup \{h_i^l(a) : a \in I_i^l,\ i < \lambda,\ l < 2\}$. Let $I_i^2 = \{a \in I_i : g_i^{-1}(a) = \eta\hat{\ }\langle\alpha + 1\rangle$ for some η, $\alpha\}$, and define $h_i^2 : I_i^2 \to I_i$ by $h_i^2(g_i(\eta\hat{\ }\langle\alpha+1\rangle)) = g_i(\eta\hat{\ }\langle\alpha\rangle)$. For every $x \in I^*$, $a = g_i(\eta) \in I_i$ we let $h_i^{<x}(a) = g_i(\eta\hat{\ }\langle\alpha\rangle)$ where α is minimal so that $g_i(\eta\hat{\ }\langle\alpha\rangle) < x$, and we define $h_i^{>x}(a)$ analogously (the functions $h_i^{<x}$, $h_i^{>x}$ are not always defined). Let $I^* = \bigcup_{\alpha<\mathrm{cf}\,\mu} I_\alpha^*$, $|I_\alpha^*| < \mu$, I_α^* increasing and continuous, so that $I \cap I_\alpha^*$ contains $\{g_i(\langle\ \rangle) : i < \lambda\}$ and is closed under h_i^2, $i < \lambda$, and under $h_i^{<x}$, $h_i^{>x}$ for $i < \lambda$, $x \in I_\alpha^*$, and $I_\alpha^* = (I \cap I_\alpha^*) \cup \{h_i^l(a) : a \in I_i^l \cap I_\alpha^*,\ i < \lambda,\ l < 2\}$.

Now for each $b \in \mathbf{I}_\alpha^* - \mathbf{I}$, we define a Dedekind cut of $\mathbf{I}_\alpha^*$: if b is the small (big) member in a pair which we inserted in a Dedekind cut of $\mathbf{I}$, then

$$D_b^\alpha = \langle\{c \in \mathbf{I}_\alpha^* : c < b\}, \{c \in \mathbf{I}_\alpha^* : c \geqq b\}\rangle$$

(then $D_b^\alpha = \langle\{c \in \mathbf{I}_\alpha^* : c \leqq b\}, \{c \in \mathbf{I}_\alpha^* : c > b\}\rangle$).

CRUCIAL FACT. Every $a \in \mathbf{I}^* - \mathbf{I}_\alpha^*$ realizes some D_b^α, $b \in \mathbf{I}_\alpha^* - \mathbf{I}$.

To prove this, assume first that $a \in (\mathbf{I}^* - \mathbf{I}_\alpha^*) \cap \mathbf{I}$. So let $a = g_i(\eta)$, and let ν be the maximal initial segment of η such that $g_i(\nu) \in I_\alpha^*$. We distinguish between two cases: (i) For no $\alpha > \eta(l(\nu))$ does $g_i(\nu\hat{\ }\langle\alpha\rangle) \in I_\alpha^*$ — in this case we take $b = h_i^0(g_i(\nu))$; we leave the checking here and in the sequel to the reader, mentioning only that $h_i^{<x}$, $h_i^{>x}$ should be used for would-be counterexamples x. (ii) If (i) is not the case, let $\alpha > \eta(l(\nu))$ be minimal such that $g_i(\nu\hat{\ }\langle\alpha\rangle) \in I_\alpha^*$. Since $I \cap I_\alpha^*$ is closed under h_i^2 and $g_i(\nu\hat{\ }\langle\eta(l(\nu))\rangle) \notin I_\alpha^*$, α must be a limit ordinal, and we take $b = h_i^1(g_i(\nu\hat{\ }\langle\alpha\rangle))$.

Assume now that $a \in (\mathbf{I}^* - \mathbf{I}_\alpha^*) - \mathbf{I}$. Then for some $l < 2$, $i < \lambda$, $a' \in I_i^l - I_\alpha^*$, $a = h_i^l(a')$. Let $a' = g_i(\eta)$, let ν be the minimal initial segment of η such that $g_i(\nu) \notin I_\alpha^*$, and let $a'' = g_i(\nu)$. One can verify that a and a'' realize the same Dedekind cut of $\mathbf{I}_\alpha^*$; thus by the first part of the proof we finish.

Now we define by induction on $\alpha \leqq \mathrm{cf}\,\mu$, J_α, F_1^α, F_2^α, g_α such that:

(1) J_α is a standard set of sequences of ordinals.

(2) F_1^α, F_2^α are as in Definition 6.4(1), except that in (b) we require just "only if".

(3) $g_\alpha : I^*_\alpha \to J_\alpha$ is one-to-one; if $a \in \mathbf{I}^*_\alpha \cap \mathbf{I}$ then $F^\alpha_1(g_\alpha(a)) = 0$; if $a \in \mathbf{I}^*_\alpha - \mathbf{I}$ and is small (big) in its pair then $F^\alpha_1(g_\alpha(a)) = 2$ (3).

(4) g_α is order preserving, where the order on range (g_α) is defined as in 6.4(3) with the additional rule that if $\eta \lessdot \nu$ then $F^\alpha_1(\eta) = 2 \Rightarrow \eta > \nu$ and $F^\alpha_1(\eta) = 3 \Rightarrow \eta < \nu$.

(5) J_α, F^α_1, F^α_2, g_α increase with α and are continuous.

For the first step of the construction, we use a standard λ-representation of $\mathbf{I}^*_0$ (it exists, since it is easily seen from the definitions of the h^l_i's that $\mathbf{I}^*_\alpha \in \mathfrak{M}_\lambda$, and $|I^*_\alpha| < \mu$), and modify F_1 so as to satisfy (3). At limit steps we take the unions. Let us describe the construction for $\alpha + 1$. For each $a \in \mathbf{I}^*_{\alpha+1} - \mathbf{I}^*_\alpha$, we pick some $b \in \mathbf{I}^*_\alpha - \mathbf{I}$ such that a realizes D^α_b. Now for every such b we take a standard λ-representation of the set of a's for which b was picked, and add it to J_α above $g_\alpha(b)$ (renaming the new nodes so that we will obtain a standard $J_{\alpha+1}$). Here again we modify F_1 of the representation so as to satisfy (3). Our definitions and construction assure that the resulting $g_{\alpha+1}$ is order preserving.

Thus, the construction goes through, and from $J_{\mathrm{cf}\,\mu}$, $F^{\mathrm{cf}\,\mu}_1$, $F^{\mathrm{cf}\,\mu}_2$ one can easily obtain a standard λ-representation of $\mathbf{I}$, with $g_{\mathrm{cf}\,\mu} \restriction I$ giving the isomorphism.

6.7. CONCLUSION. For any $\lambda \geqq \aleph_0$ and beautiful $\kappa > \lambda$, $\mathfrak{M}_\lambda$ is $[\kappa]$-I-bqo. Moreover, e.g., if $\mathbf{Q}$ is $[\kappa; 2^\lambda]$-X-bqo, then $\mathfrak{M}_\lambda[\mathbf{Q}]$ is $[\kappa; 2^\lambda]$-X-bqo, where $\mathfrak{M}_\lambda[\mathbf{Q}]$ is the class of $(\mathbf{I}, f)$, $\mathbf{I} \in \mathfrak{M}_\lambda$, $f : \mathbf{I} \to \mathbf{Q}$, ordered by embeddability where $F : (\mathbf{I}_1, f_1) \to (\mathbf{I}_2, f_2)$ is an embedding if $F : \mathbf{I}_1 \to \mathbf{I}_2$ is order preserving and $f_1(x) \leqq f_2[F(x)]$.

PROOF. By 6.5, 6.6, 5.3.

REMARK. (1) This is a quite strong result. We can prove by it $[\kappa; \lambda]$-I-bqo, e.g., for $(\mathcal{P}(Q), \leqq_1)$ (the mappings are one-to-one), and the trees with the embeddings Nash-Williams used.

(2) For $\lambda = \aleph_0$, we could get $\kappa = \aleph_0$, which is the celebrated result of Laver, which we generalize here.

6.8. THEOREM. *Let* $\mathbf{Q}$ *be a quasi-order,* λ *a regular cardinal. Then we can find a function* H*, such that*

(1) $\mathrm{Dom}\, H = \{a \in \mathcal{P}^0_{<\infty}(\mathbf{Q}) : |Tc(a)| < \lambda\}$,

(2) $\mathrm{Range}\, H \subseteq \mathrm{Seq}_{<\lambda^+}(\mathbf{Q})$,

(3) *if not* $a \leqq b$ *then not* $H(a) \leqq H(b)$.

REMARK. $\mathrm{Seq}_{<\alpha}(\mathbf{Q})$ is ordered by $\langle q_i : i < \alpha \rangle \leqq \langle q^j : j < \beta \rangle$ iff there is a monotonic increasing $h : \alpha \to \beta$, $q_i \leqq q^{h(i)}$.

PROOF. We define $H(a)$ by induction on $\alpha = \mathrm{Dp}(a) = \min\{\beta : a \in \mathcal{P}^0_\beta(\mathbf{Q})\}$. For $0 < i < \lambda$, let $f_i : \lambda \to i$ be such that for every $j < i$, $\{\alpha < \lambda : f_i(\alpha) = j\}$ has power λ.

$\alpha = 0$. So $a \in \mathbf{Q}$, and we let $H(a) = \langle a : i < \lambda \rangle$.

$\alpha > 0$. Let $a = \{a_i : i < i(a) < \lambda\}$, so $\mathrm{Dp}(a_i) < \alpha$, and let

$$H(a) = H(a_{f_{i(a)}(0)})^\wedge H(a_{f_{i(a)}(1)})^\wedge \cdots {}^\wedge H(a_{f_{i(a)}(\alpha)})^\wedge \cdots \qquad (\alpha < \lambda).$$

Clearly $H(a)$ has always length an ordinal of cofinality λ, and in fact $\lambda^{\mathrm{Dp}(a)+1}$.

We prove by induction on α that:

(*) If $a, b \in \mathcal{P}^0_{<\infty}(Q)$, $\mathrm{Dp}(a)$, $\mathrm{Dp}(b) \leqq \alpha$, and not $a \leqq b$ then no end segment of $H(a)$ can be embedded into $H(b)$ (when they are defined). Formally, for no $\gamma < \lambda^{\mathrm{Dp}(a)+1}$ is there a function $f : [\gamma, \lambda^{\mathrm{Dp}(a)+1}) \to \lambda^{\mathrm{Dp}(b)+1}$, f strictly increasing and $\mathbf{Q} \models H(a)(i) \leqq H(b)(f(i))$ for every $i \in [\gamma, \lambda^{\mathrm{Dp}(a)+1})$.

This is straightforward.

6.9. CONCLUSION. The X-well-ordering number of $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ is at least that of $\mathcal{P}^0_{<\infty}(\mathbf{Q})$ and at most that of $\mathcal{T}^2(\mathbf{Q})$.

REMARK. Many times the lower and upper bound agree (use 2.5 and 5.3), so we get an exact value.

6.10. CLAIM. If the D-well-ordering number of $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ is a strong limit $\kappa > \aleph_0$ *then* the I-well-ordering number of it is also κ.

PROOF. Clearly the I-well-ordering number of $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ is $\leqq \kappa$; if it is $\lambda < \kappa$, there are $\bar{q}_i$ $(i < (2^\lambda)^+)$, $\bar{q}_i \not\leqq \bar{q}_j$ for $i < j$. By the Erdos–Rado theorem, w.l.o.g. $\langle l(\bar{q}_i) : i < \lambda \rangle$ is strictly decreasing, strictly increasing or constant. The first case is impossible; the second implies $\{\bar{q}_i : i < \lambda\}$ are pairwise incomparable. So assume $l(\bar{q}_i) = \alpha$ for $i < \lambda$.

We can assume that α is divisible by λ (otherwise replace $\bar{q}_i = \langle q_{i,j} : j < \alpha \rangle$ by $\langle q'_{i,j} : j < \lambda\alpha \rangle$, $q'_{i,\lambda j+\gamma} = q_{i,j}$ for $j < \alpha$, $\gamma < \lambda$) and then choose $q' \in \mathbf{Q}$, let $\bar{q}'_i = \langle q' : j < i \rangle$, and let $\bar{q}^*_i = \bar{q}_i{}^\wedge \bar{q}'_i$.

Now $\{\bar{q}^*_i : i < \lambda\} \subseteq \mathrm{Seq}_{<\infty}(\mathbf{Q})$ are pairwise incomparable.

REMARK. Really Erdös–Rado is not needed, and for $\lambda < \kappa$, λ regular the I-well-ordering number of $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ is $> \lambda$.

We return now to the computation of the well-ordering number of $\mathfrak{M}_\lambda$.

6.11. LEMMA. *Suppose $\kappa_0 < \kappa_1$ are beautiful cardinals, but there is no beautiful cardinal κ, $\kappa_0 < \kappa < \kappa_1$.*

If $\aleph_1 + \kappa_0 \leqq \lambda < \kappa_1$, *then the well-ordering number of* $\mathfrak{M}_\lambda$ *is* κ_1. *Moreover for any* $\kappa_0 \leqq \mu < \kappa_1$, *there are orders* $\mathbf{I}_\alpha$ $(\alpha < \mu)$ *such that*

(i) $|I_\alpha| = \mu^+$,

(ii) $\alpha \neq \beta$ *implies that not* $\mathbf{I}_\alpha \leqq \mathbf{I}_\beta$,

(iii) $\mathbf{I}_\alpha$ *is the union of* λ *well-orderings (so even of* $\aleph_1 + \kappa_0$ *well-orderings).*

Proof. By [24] there are $\aleph_1 + \kappa_0$ orderings $\mathbf{J}_i$ $(i < \aleph_1 + \kappa_0)$ each with no first element of power $\aleph_1 + \kappa_0$, no initial segment of one embeddable in a well-ordered sum of copies of the others.† Let $\kappa_0 \leqq \mu < \kappa_1$. We consider $\mathbf{Q} = (\aleph_1 + \kappa_0, =)$. Checking the proof of 2.5, one can see that it holds for $\mathscr{P}^0_\alpha$ too. Since by our assumptions not $\mu \xrightarrow{w} (\omega)^{<\omega}_{\kappa_0}$, it follows that in some $\mathscr{P}^0_\alpha(\mathbf{Q})$ there are μ pairwise incomparable elements. By 4.10 we may assume that they are of hereditary power $< \mu^+$, so that applying the function H of 6.8 we obtain a pairwise incomparable family $\{f_\alpha\}_{\alpha<\mu} \subseteq \mathrm{Seq}_{<\infty}(\mathbf{Q})$ such that the length of f_α is an ordinal β_α, $\mu^+ \leqq \beta_\alpha < \mu^{++}$. We let $\mathbf{I}_\alpha = \Sigma_{j<\beta_\alpha}[\mathbf{J}_{2f_\alpha(j)} + \mathbf{J}_{2f_\alpha(j)+1}]$. Requirement (i) is obviously satisfied, and for (iii) enumerate each summand (each has power $\aleph_1 + \kappa_0$) and consider for every $\gamma < \aleph_1 + \kappa_0$ the set of γ-th elements. Finally if g embeds $\mathbf{I}_\alpha$ into $\mathbf{I}_{\alpha'}$, then for $j < \beta_\alpha$ we let j' be such that g'' maps an initial segment of $\mathbf{J}_{2f_\alpha(j)}$ into $\mathbf{J}_{2f_{\alpha''}(j'')}$ hence necessarily $f_\alpha(j) = f_{\alpha'}(j')$. Then the mapping $j \to j'$ shows that $f_\alpha \leqq f'_\alpha$, thus proving (ii).

6.12. Conclusion. The well-ordering number of $\mathfrak{M}_\lambda$ is the first beautiful cardinal $> \lambda$, provided that $\lambda > \aleph_0$.

§7. Some exact computation

7.1. Definition. (1) $\mathscr{T}^{-2}_{<\infty}(\mathbf{Q})$ is the class of $(\mathbf{T}, f) \in \mathscr{T}^0_{<\infty}(\mathbf{Q})$ ordered by embeddings $F : (\mathbf{T}_1, f_1) \to (\mathbf{T}_2, f_2)$ preserving $<$ and satisfying $f_1(t) \leqq f_2(F(t))$.

(2) $\mathscr{T}^{-1}_{<\infty}(\mathbf{Q})$ is defined similarly but F must preserve also the relation "t is the largest lower bound of s_1 and s_2" (this is the order Nash-Williams used).

(3) $\mathscr{T}^l(\mathbf{Q})$, $\mathscr{T}^l_{\leqq\alpha}(\mathbf{Q})$ $(l = -1, -2)$ are defined similarly.

7.2. Definition. Let $\mathbf{Q}$ be a quasi-order, α an ordinal. We define the quasi-order $\mathscr{P}^1_\alpha(\mathbf{Q})$ as follows: the elements are as in Definition 1.3(3), but we

† Let $\lambda = \aleph_1 + \kappa_0$. If $\lambda > \aleph_0$ regular or $\lambda = \lambda^{\aleph_0}$, trivially by [24]. The remaining case is λ strong limit of cofinality $\aleph_0$. Let $\lambda = \Sigma_{n<\omega}\lambda_n$, $\lambda_n < \lambda_{n+1}$, λ_n regular. For each λ_n let S_α $(\alpha < \lambda_n)$ be pairwise disjoint subsets of $\{\delta : \delta < \lambda_n, \mathrm{cf}\,\delta = \aleph_0\}$, for $\delta \in \bigcup S_\alpha$ let η_δ be an increasing ω-sequence of ordinals with limit δ, let $\mathbf{I}^n_\alpha = {}^{\omega>}\lambda_n \vee \{\eta_\delta : \delta \in S_\alpha\}$ ordered lexicographically. Let $\mathbf{J}_n$ $(n < \omega)$ be sets of reals, $|\mathbf{J}_n| = 2^{\aleph_0}$ and let $\mathbf{J}^n_\alpha$ be the inverse of $\mathbf{I}^n_\alpha \times \mathbf{J}_n$. Then $\{\mathbf{J}^n_\alpha : \alpha < \lambda_n, n < \omega\}$ is as required, if the $\mathbf{J}_n$ were chosen correctly.

allow sets with repetitions and omit the empty sets; to define the order, we do the following modifications in 1.3(4): in (b)(i) the function has to be one-to-one, (b)(ii) is omitted, and in (b)(iii) A_1 has to be in Q.

7.3. LEMMA. *If* $\mathbf{Q}$ *is* $\aleph_0$-I-bqo *then so are* $\mathcal{T}^{-1}(\mathbf{Q})$, $\mathcal{T}^{-2}(\mathbf{Q})$, $\mathcal{P}^{**}_{<\infty}(\mathbf{Q})$, $\mathcal{P}^{1}_{<\infty}(\mathbf{Q})$, $\mathrm{Seq}_{<\infty}(\mathbf{Q})$.

PROOF. By Nash-Williams.

7.4. DEFINITION. Let $\mathbf{Q}_1$, $\mathbf{Q}_2$ be quasi-orders (their universes may be proper classes). We say that $\mathbf{Q}_1$ is locally embeddable into $\mathbf{Q}_2$ if for every *set* K, $K \subseteq Q_1$, there is a function $H : K \to Q_2$ satisfying: $H(q) \leqq H(q') \Rightarrow q \leqq q'$. We say that $\mathbf{Q}_1$ is embeddable into $\mathbf{Q}_2$ if there is such a function with domain Q_1.

7.5. CLAIM. Suppose $\mathbf{Q}_1$ is locally embeddable into $\mathbf{Q}_2$. If $\mathbf{Q}_2$ is κ-X-well-ordered, then $\mathbf{Q}_1$ is κ-X-well-ordered. Similarly for "$\mathbf{Q}$ is $[\kappa,\alpha;\lambda]$-X-bqo", "$\mathbf{Q}\times(\omega,\leqq)$ is κ-X-bqo", etc.

PROOF. Trivial.

REMARK. We have already proved some local embeddability results.

7.6. LEMMA. $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ *is embeddable into* $\mathcal{T}^{-1}_{<\infty}(\mathbf{Q})$ *and* $\mathcal{T}^{-1}(\mathbf{Q})$ *which are embeddable into* $\mathcal{T}^{0}(\mathbf{Q})$.

PROOF. Trivial.

7.7. LEMMA. (1) $\mathcal{T}^{-2}(\mathbf{Q})$ *is locally embeddable into* $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$.
(2) *Similarly for* $\mathcal{T}^{-1}(\mathbf{Q})$.

PROOF. Like 5.4(4).

7.8. LEMMA. $\mathrm{Seq}_{<\infty}(\mathbf{Q})$ *is embeddable into* $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$.

PROOF. For any $\bar{q} = \langle q_i : i < \alpha\rangle \in \mathrm{Seq}_{<\infty}(\mathbf{Q})$ let $\mathbf{T}_{\bar{q}} = \{\eta : \eta$ a decreasing sequence of ordinals $< \alpha\}$, $f_{\bar{q}}(\langle\ \rangle) =$ arbitrary, $f_{\bar{q}}(\langle\eta(0),\cdots,\eta(n-1)\rangle) = q_{\eta(n-1)}$.

Now, if F embeds $(\mathbf{T}_{\bar{q}}, f_{\bar{q}})$ into $(\mathbf{T}_{\bar{r}}, f_{\bar{r}})$ then the function F^* defined by $F^*(i) = \min\{j$: for some $\eta \in \mathbf{T}_{\bar{q}}$, $\eta(l(\eta)-1) = i$ and $(F(\eta))(l(F(\eta))-1) = j\}$ shows that $\bar{q} \leqq \bar{r}$.

7.9. LEMMA. (1) $\mathcal{P}^{**}_{<\infty}(\mathbf{Q})$ *can be locally embedded into* $\mathrm{Seq}_{<\infty}(\mathbf{Q})$.
(2) $\mathcal{P}^{0}_{<\infty}(\mathbf{Q})$ *can be embedded into* $\mathcal{P}^{**}_{<\infty}(\mathbf{Q})$.

PROOF. (1) By 6.8's proof.
(2) Trivial.

7.10. LEMMA. *$\mathcal{T}^{0}_{<\infty}(\mathbf{Q})$ is locally embeddable into $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ if $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-well-ordered.*

PROOF. We observe that if $|\mathbf{T}|<\lambda$, $\mathbf{T}\in\mathcal{T}^{0}_{<\infty}$, λ regular, then

(i) $\mathrm{Dp}(\mathbf{T})<\lambda$,

(ii) $\mathbf{T}$ is isomorphic to a tree of finite sequences of ordinals $<\lambda$ closed under initial segments.

Let H be a subset of $\mathcal{T}^{0}_{<\infty}(\mathbf{Q})$ (which we want to embed into $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$), and let $\langle(\mathbf{T}^n,f^n):n<\omega\rangle$ exemplify that $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered. We choose a regular λ bigger than the cardinalities of all the trees in H and the $\mathbf{T}^n$'s. We also choose $q^*,q^{**}\in\mathbf{Q}$ such that $q^*\not\leq q^{**}$ (this is possible since otherwise $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ must be $\aleph_0$-I-well-ordered).

Given $(\mathbf{T},f)\in H$, we define $(\bar{\mathbf{T}},\bar f)$ as follows: $\bar{\mathbf{T}}=\mathbf{T}\cup\mathbf{T}'\cup\mathbf{T}''$, where: $\mathbf{T}'=\{\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau:\eta\in\mathbf{T},\ \tau$ a decreasing sequence of ordinals $<\lambda\}$, $\mathbf{T}''=\{\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\hat{\ }\langle\lambda\rangle\hat{\ }\tau':\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\in\mathbf{T}',\ \tau'\in\mathbf{T}^{l(\eta)}\}$;

$$\bar f(\nu)=\begin{cases} f(\nu), & \nu\in\mathbf{T},\\ q^{**}, & \nu\in\mathbf{T}',\ \nu(l(\nu)-1)\neq 0,\\ q^{*}, & \nu\in\mathbf{T}',\ \nu(l(\nu)-1)=0,\\ f^{l(\eta)}(\tau'), & \nu=\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\hat{\ }\langle\lambda\rangle\hat{\ }\tau'\in\mathbf{T}''.\end{cases}$$

It is easily verified that:

$$(\bar{\mathbf{T}},\bar f)\in\mathcal{T}^{-2}_{<\infty}(\mathbf{Q});$$

$$\nu=\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\hat{\ }\langle\lambda\rangle\in\mathbf{T}''\Rightarrow \mathrm{Dp}_{\bar{\mathbf{T}}}(\nu)=\mathrm{Dp}(\mathbf{T}^{l(\eta)})<\lambda;$$

$$\nu\in\mathbf{T}',\ \nu(l(\nu)-1)=\alpha<\lambda\Rightarrow\alpha\leqq\mathrm{Dp}_{\bar{\mathbf{T}}}(\nu)<\lambda;$$

$$\eta\in\mathbf{T}\Rightarrow\mathrm{Dp}_{\bar{\mathbf{T}}}(\eta\hat{\ }\langle\lambda\rangle)=\lambda,\ \mathrm{Dp}_{\bar{\mathbf{T}}}(\eta)>\lambda.$$

Now, assume that $F:(\bar{\mathbf{T}}_1,\bar f_1)\to(\bar{\mathbf{T}}_2,\bar f_2)$ is a $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ embedding. Since always $\mathrm{Dp}_{\bar{\mathbf{T}}_1}(\nu)\leqq\mathrm{Dp}_{\bar{\mathbf{T}}_2}(F(\nu))$, we conclude that the restriction of F to $\mathbf{T}_1$ maps it into $\mathbf{T}_2$, so that in order to prove that $(\mathbf{T}_1,f_1)\leqq(\mathbf{T}_2,f_2)$ in $\mathcal{T}^{0}_{<\infty}(\mathbf{Q})$ it suffices to show that $l(F(\eta))>l(\eta)$ never occurs for $\eta\in\mathbf{T}_1$.

Assume that $\eta\in\mathbf{T}_1$ is a counter-example to this. If for all $\nu=\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\in\mathbf{T}_1'$, $F(\nu)\in\mathbf{T}_2$ then for all such ν, $\mathrm{Dp}_{\bar{\mathbf{T}}_1}(\nu)\leqq\mathrm{Dp}_{\mathbf{T}_2}(F(\nu))+1+\mathrm{Dp}(\mathbf{T}^{l(\eta)})$ (prove first that the depth of ν in $\mathbf{T}_1\cup\mathbf{T}_1'$ is $\leqq\mathrm{Dp}_{\mathbf{T}_2}(F(\nu))$ and then by induction on $\mathrm{Dp}_{\bar{\mathbf{T}}_1}(\nu)$), in particular $\mathrm{Dp}_{\bar{\mathbf{T}}_1}(\eta\hat{\ }\langle\lambda\rangle)<\lambda$, a contradiction. So for some $\nu=\eta\hat{\ }\langle\lambda\rangle\hat{\ }\tau\in\mathbf{T}_1'$, $F(\nu)\in\mathbf{T}_2'\cup\mathbf{T}_2''$; by considering some extension, if necessary, we may assume that $\nu(l(\nu)-1)=0$. Since $\bar f_1(\nu)\leqq\bar f_2(F(\nu))$, either $F(\nu)\in\mathbf{T}_2'$ and $(F(\nu))(l(F(\nu))-1)=0$ or $F(\nu)\in\mathbf{T}_2''$. In any case, F induces a $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ embedding of $(\mathbf{T}^{l(\eta)},f^{l(\eta)})$ into $(\mathbf{T}^n,f^n)$ for some $n\geqq l(F(\eta))>l(\eta)$, a contradiction.

7.11. CLAIM. In 7.10 it suffices to demand that $\mathbf{Q}$ is not $\aleph_0$-I-bqo.

PROOF. By 7.9(2), 7.9(1), 7.8 (and the transitivity of local embeddability), $\mathcal{P}^0_{<\infty}(\mathbf{Q})$ is locally embeddable into $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$. Now if $\mathbf{Q}$ is not $\aleph_0$-I-bqo, $\mathcal{P}^0_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered (check in the proof of 1.12), hence $\mathcal{T}^{-2}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered.

7.12. DEFINITION. $\mathcal{T}^{-3}_{<\infty}(\mathbf{Q})$ is the class of pairs $(\mathbf{T}, f)$, $\mathbf{T}$ as usual and $f: \{\eta \in \mathbf{T} : \mathrm{Dp}_{\mathbf{T}}(\eta) = 0\} \to Q$. We call $F : (\mathbf{T}_1, f_1) \to (\mathbf{T}_2, f_2)$ an embedding if it is a function from $\mathbf{T}_1$ to $\mathbf{T}_2$ preserving $<$ and satisfying: $\eta \in \mathrm{Dom}\, f_1 \Rightarrow [F(\eta) \in \mathrm{Dom}\, f_2$ and $f_1(\eta) \leqq f_2(F(\eta))]$.

Notice that $(\mathbf{T}_1, f_1) \leqq (\mathbf{T}_2, f_2)$ in $\mathcal{T}^{-3}_{<\infty}(\mathbf{Q})$ iff there is a function F from $\mathbf{T}_1$ to $\mathbf{T}_2$ preserving $<$ and satisfying: for every $\eta \in \mathrm{Dom}\, f_1$ there is ν, $F(\eta) \leqq \nu \in \mathrm{Dom}\, f_2$, $f_1(\eta) \leqq f_2(\nu)$.

We define $\mathcal{T}^{-3}(\mathbf{Q})$, $\mathcal{T}^{-3}_{\leqq\alpha}(\mathbf{Q})$ similarly.

7.13. LEMMA. *$\mathcal{P}^{**}_{\alpha}(\mathbf{Q})$, $\mathcal{T}^{-3}_{\leqq\alpha}(\mathbf{Q})$ are isomorphic.*

PROOF. Straightforward.

7.14. DEFINITION. $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$ is the class of pairs $(\mathbf{T}, f)$, $\mathbf{T}$ as usual and f a function from $\mathbf{T}$ into $Q \cup$ "the ordinals", such that $\mathrm{Dp}_{\mathbf{T}}(\eta) = 0$ implies $f(\eta) \in Q$, $\mathrm{Dp}_{\mathbf{T}}(\eta) > 0$ implies $f(\eta)$ is an ordinal and $\eta \lessdot \nu < \tau \in \mathbf{T}$ implies $f(\eta) < f(\nu)$. $F : (\mathbf{T}_1, f_1) \to (\mathbf{T}_2, f_2)$ is an embedding if F maps $\mathbf{T}_1$ into $\mathbf{T}_2$, preserves $<$ and the depth being 0, and satisfies $f_1(\eta) \leqq f_2(F(\eta))$ (in $\mathbf{Q}$ if $\mathrm{Dp}_{\mathbf{T}}(\eta) = 0$, as ordinals otherwise); w.l.o.g. no ordinal belongs to Q.

NOTATION. For $(\mathbf{T}, f) \in \mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$, $\eta \in \mathbf{T}$, let (we consider $\mathbf{T}$ as a tree of sequences closed under initial segments):

$$\mathbf{T}_{(\eta)} = \{\nu : \eta^\wedge \nu \in \mathbf{T}\},$$

$$f_{(\eta)}(\nu) = f(\eta^\wedge \nu) \qquad \text{for } \eta^\wedge \nu \in \mathbf{T},$$

$$(\mathbf{T}, f)_{(\eta)} = (\mathbf{T}_{(\eta)}, f_{(\eta)}).$$

7.15. LEMMA. *If $\mathbf{Q}$ is not $\aleph_0$-I-bqo, then $\mathcal{T}^0_{<\infty}(\mathbf{Q})$ can be locally embedded into $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q} \times \mathbf{Q})$.*

PROOF. Basically we repeat the proof of 7.10, 7.11. As in 7.11, we know that $\mathcal{P}^0_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered, hence surely $\mathcal{P}^{**}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered, which by 7.13 means that $\mathcal{T}^{-3}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered, hence surely $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$ is not $\aleph_0$-I-well-ordered. Now we use an example of the latter to define a local embedding as in 7.10, with the following differences: for $\nu \in \mathbf{T}$ and $\nu \in \mathbf{T}'$ such that $\nu(l(\nu)-1)\neq 0$ we let $\bar{f}(\nu)$ be ordinals chosen as small as possible (respecting the order of the tree); for $\nu\in\mathbf{T}'$, $\nu(l(\nu)-1)=0$ we let $\bar{f}(\nu)=\gamma$ be a fixed ordinal which is bigger than all those chosen in the previous stage for any of the trees; for $\nu=\eta^\wedge\langle\lambda\rangle^\wedge\tau^\wedge\langle\lambda\rangle^\wedge\tau'\in\mathbf{T}''$, if $\mathrm{Dp}_{\bar{\mathbf{T}}}(\nu)>0$ then $\bar{f}(\nu)=\gamma+1+f^{l(\eta)}(\tau')$ and if $\mathrm{Dp}_{\bar{\mathbf{T}}}(\nu)=0$ then $\bar{f}(\nu)=(f(\eta),f^{l(\eta)}(\tau'))$. The reader will see that the proof of 7.10 goes through; in addition to what was done there, we have to verify that $f_1(\eta)\leqq f_2(F(\eta))$ for $\eta\in\mathbf{T}_1$, and for this one has to notice that in order to avoid the contradiction in the end of the proof of 7.10, it is necessary that $n=l(F(\eta))=l(\eta)$.

7.16. Lemma. *$\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q}\times\mathbf{Q})$ is locally embeddable into $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$.*

Proof. Let $K\subseteq\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q}\times\mathbf{Q})$ be a subset, and let γ be an ordinal bigger than all those occurring in K. Given $(\mathbf{T},f)\in K$, we obtain $(\bar{\mathbf{T}},\bar{f})\in\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$ by the following modifications: if $f(\eta)=(q_0,q_1)$ we let $\bar{f}(\eta)=\gamma$, we add new nodes η_i, $i=0,\cdots,4$, so that $\eta<\eta_2<\eta_3<\eta_1$ and $\eta<\eta_4<\eta_0$, letting $\bar{f}(\eta_i)=\gamma+i$ for $i=2,3,4$ and $\bar{f}(\eta_i)=q_i$ for $i=0,1$. It is straightforward to check that $(\mathbf{T},f)\rightarrow(\bar{\mathbf{T}},\bar{f})$ is indeed a local embedding of $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q}\times\mathbf{Q})$ into $\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$.

7.17. Lemma. *$\mathcal{T}^{-2.5}_{<\infty}(\mathbf{Q})$ is embeddable into $\mathcal{P}^{1}_{<\infty}(\mathbf{Q})$.*

Proof. We define the embedding H by induction on $\mathrm{Dp}(\mathbf{T})$. If $\mathrm{Dp}(\mathbf{T})=0$ then $f(\langle\ \rangle)=q\in Q$ and we let $H(\mathbf{T},f)=q$. If $\mathrm{Dp}(\mathbf{T})>0$ we let $H(\mathbf{T},f)=\{\{H((\mathbf{T},f)_{\langle\eta\rangle}):l(\eta)=1\}:\aleph_{f(\langle\ \rangle)}\text{ times}\}$ (this is a set with repetitions).

7.18. Lemma. *$\mathcal{P}^{1}_{<\infty}(\mathbf{Q})$ is embeddable into $\mathcal{T}^{1}_{<\infty}(\mathbf{Q})$.*

Proof. We define the embedding H by induction on $\mathrm{Dp}(a)=\min\{\alpha:a\in\mathcal{P}^{1}_{\alpha}(\mathbf{Q})\}$. If $\mathrm{Dp}(a)=0$ we let $H(a)$ be a tree with a single node labeled a. If $\mathrm{Dp}(a)>0$ we let $H(a)$ be a tree constructed from copies of the $H(b)$'s, b ranging over the elements of a, so that each node in level 1 of $H(a)$ corresponds to one such b (being the root of a copy of $H(b)$); we choose arbitrarily a well-ordering of level 1, and also an element of Q among the labels in this level to label the root of $H(a)$.

7.19. Theorem. *Suppose $\mathbf{Q}$ is not $\aleph_0$-I-bqo. Then the following conditions on κ are equivalent for $X\in\{I,D\}$, $F\in\{\mathcal{T}^2,\mathcal{T}^2_{<\infty},\mathcal{T}^1,\mathcal{T}^1_{<\infty},\mathcal{T}^0,\mathcal{T}^0_{<\infty},\mathcal{T}^{-1},\mathcal{T}^{-1}_{<\infty},\mathcal{T}^{-2},\mathcal{T}^{-2}_{<\infty},\mathcal{T}^{-2.5}_{<\infty},\mathcal{P}^{1}_{<\infty}\}$:*

(1) *$\dot{\mathbf{Q}}$ is $[\kappa;\aleph_0]$-bqo.*

(2) *$F(\mathbf{Q})$ is κ-X-bqo.*

(3) *$F(\mathbf{Q})$ is κ-X-well-ordered.*

REMARK. As (1) does not involve X or F, we get all variants are equivalent.

PROOF. If F is one of the $\mathcal{T}^l$ or $\mathcal{T}^l_{<\alpha}$, $l \geqq 0$, use the results in §5. Now, using the results of this section, one can see that the theorem holds also for the other F's listed.

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